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ELECTRONICS AND ELECTRICAL ENGINEERING

RADAR SIGNAL RECEPTION AGAINST BACKGROUND OF RANDOM INTERFERENCE

By

S. Ye. Fal'kovich

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USSR REPORT

ELECTRONICS AND ELECTRICAL ENGINEERING

RADAR SIGNAL RECEPTION AGAINST BACKGROUND OF RANDOM INTERFERENCE

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CONTENTS

[Text] Chapter 1. Definitions and Initial Formulas

1.1. Some Definitions and Formulas for Determinate Functions

The determinate, i.e., random, functions or temporal processes з(t) examined in the present section serve as the analytical representation of radar signals.

We shall employ the symbol $\leftrightarrow$ to designate a pair of Fourier transforms, e.g.

$$s(t) \leftrightarrow g(f), \tag{1.1}$$

which in expanded form means

$$s(t) = \int_{-\infty}^{\infty} g(f)\, e^{j2\pi ft} df,$$
$$g(f) = \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi ft} dt. \tag{1.2}$$

The letters t and f designate time and frequency. Here and below we shall not stipulate the existence of improper integrals, since it is assumed that all of the integrals of the determinate functions examined are absolutely convergent.

The real temporal process $\mathfrak{s}(t)$, representing high frequency oscillation with slowly varying amplitude S(t) and phase $2\pi\theta(t)$

$$\mathfrak{s}(t) = S(t)\cos\{2\pi f_0 t + 2\pi\theta(t) + \varphi\} \tag{1.3}$$

can be written in the form

$$\mathfrak{s}(t) = \mathrm{Re}\{\dot{S}(t)\, e^{j[2\pi f_0 t + \varphi]}\} \tag{1.4}$$

or in symbolic form

$$\dot{\mathfrak{s}}(t) = \dot{S}(t)\, e^{j[2\pi f_0 t + \varphi]}, \tag{1.5}$$

where f_0 -- carrier frequency, ϕ -- initial phase, S(t) -- complex amplitude(envelope)

$$\dot{S}(t) = S(t)\, e^{j2\pi\theta(t)}, \tag{1.6}$$

which determines the form of the signal or the nature of the modulation. Accordingly, the function S(t) can also be called the modulation function.

If the spectral density of the function S(t), called the modulation spectrum below, is G(f)

$$G(f) \leftrightarrow \dot{S}(t), \tag{1.7}$$

the frequency spectrum of the process $\mathfrak{s}(t)$ and its symbolic representation $\dot{\mathfrak{s}}(t)$ will accordingly be

$$\mathfrak{s}(t) \leftrightarrow g(f) = \frac{1}{2} G(f - f_0)\, e^{j\varphi} + \frac{1}{2} \overline{G(-f - f_0)}\, e^{-j\varphi} \tag{1.8}$$

$$\dot{\mathfrak{s}}(t) \leftrightarrow \dot{g}(f) = G(f - f_0)\, e^{j\varphi}. \tag{1.9}$$

The overscribed bar designates complex conjugate quantities. When the carrier frequency is significantly greater than the modulation spectrum width, the frequency spectrum of the symbolic representation imitates the frequency spectrum of the original on the positive half-axis f (to within a constant of 2).

Frequent use is made below of the scalar product

$$\int_{-\infty}^{\infty} \text{з}_1(t)\, \text{з}_2(t)\, dt \tag{1.10}$$

of the two oscillating processes $\text{з}_1(t)$ and $\text{з}_2(t)$ with close carrier frequencies f_{01} and f_{02}

$$|f_{01} - f_{02}| \ll f_{01}, f_{02}. \tag{1.11}$$

Assuming that the result of integrating an oscillating process with slowly varying amplitude and phase over a sufficiently large interval is practically zero, the scalar product (1.10) can be represented as

$$\begin{aligned}\int_{-\infty}^{\infty} \text{з}_1(t)\, \text{з}_2(t)\, dt &= \frac{1}{2}\,\mathrm{Re}\{e^{j(\varphi_1-\varphi_2)} \int_{-\infty}^{\infty} S_1(t)\, \overline{S_2(t)}\, e^{j2\pi(f_{01}-f_{02})t} dt\} = \\ &= \frac{1}{2}\,\mathrm{Re}\{e^{j(\varphi_1-\varphi_2)} \int_{-\infty}^{\infty} G_1(f)\, \overline{G_2(f+f_{01}-f_{02})}\, df\}.\end{aligned} \tag{1.12}$$

Following the definitions introduced in [18, 8], we shall call the quantity

$$Q^2 = \int_{-\infty}^{\infty} \text{з}^2(t)\, dt = \frac{1}{2}\int_{-\infty}^{\infty} |S(t)|^2\, dt \tag{1.13}$$

the total energy, and the function

$$\begin{aligned}\psi(\tau) = \frac{1}{Q^2}\int_{-\infty}^{\infty} \text{з}(t)\, \text{з}(t-\tau)\, dt = \mathrm{Re}\,\{e^{j2\pi f_0 t} \times \\ \times \frac{1}{2Q^2}\int_{-\infty}^{\infty} S(t)\, \overline{S(t-\tau)}\, dt\}\end{aligned} \tag{1.14}$$

the correlation function of the process з(t). We note that the term correlation function employed in the theory of random processes is analogously applied here to a determinate process -- the signal з(t). We can write

$$\psi(\tau) = |\Psi(\tau)| \cos[2\pi f_0 \tau + \varkappa(\tau)], \quad (1.15)$$

where $|\Psi(\tau)|$ and $\kappa(t)$ -- are the absolute value and argument of the complex correlation modulation function $\Psi(\tau)$

$$\Psi(\tau) = \frac{1}{2Q^2} \int_{-\infty}^{\infty} S(t) \overline{S(t-\tau)}\, dt. \quad (1.16)$$

The power spectrum of the signal $\omega(f)$ is associated with the modulation power spectrum $\Omega(f)$

$$\Omega(f) = |G(f)|^2 \div 2Q^2 \Psi(\tau) \quad (1.17)$$

by this formula, which is analogous to (1.8):

$$\omega(f) = |\dot{g}(f)|^2 = \frac{1}{4} \Omega(f - f_0) + \frac{1}{4} \Omega(-f - f_0). \quad (1.18)$$

The carrier frequency is defined by the relationship [8]

$$\int_{-\infty}^{\infty} (f - f_0) \Omega(f - f_0)\, df = \int_{-\infty}^{\infty} S'(t) \overline{S(t)}\, dt = 0. \quad (1.19)$$

It can be assumed in general that the power spectrum of a radar signal is symmetrical about the carrier frequency f_0, i.e., the modulation power spectrum $\Omega(f)$ is an even function. Furthermore, the modulation correlation function $\Psi(\tau)$ is also an even real function. Accordingly,

$$\varkappa(\tau) = 0 \text{ for } \pi,$$

while (1.15) is transformed to

$$\psi(\tau)=\Psi(\tau)\cos 2\pi f_0\tau. \tag{1.20}$$

Consequently, the correlation function of a radar signal looks like oscillation having only amplitude modulation.

In some calculations it is helpful to employ an approximate representation of the modulation correlation function. This study employs the rectangular approximation

$$\Psi(\tau)=\begin{cases}1 \text{ for } & |\tau|\leqslant\frac{1}{2}\tau_\kappa, \\ 0 \text{ for } & |\tau|>\frac{1}{2}\tau_\kappa,\end{cases} \tag{1.21}$$

where τ_κ -- correlation time

$$\tau_\kappa=\int_{-\infty}^{\infty}|\Psi(\tau)|^2\,d\tau, \tag{1.22}$$

and a representation based on expanding the function $\Psi(\tau)$ by powers of the parameter τ. If we introduce the notation

$$\beta_\tau^2=-\Psi''(0)=\frac{\int_{-\infty}^{\infty}|S'(t)|^2\,dt}{\int_{-\infty}^{\infty}|S(t)|^2\,dt}=4\pi^2\frac{\int_{-\infty}^{\infty}f^2\Omega(f)\,df}{\int_{-\infty}^{\infty}\Omega(f)\,df}, \tag{1.23}$$

then for small τ

$$\Psi(\tau)=1-\frac{1}{2}\beta_\tau^2\tau^2. \tag{1.24}$$

The parameter β_τ defines the dispersion of the frequency components of the modulation spectrum $\Omega(f)$ with respect to the "center of gravity" (the point f=0), and can serve as a measure of the spectrum width of the signal з(t).

The signals emitted by a radar usually have a periodically repeating modulation function which can be written in the form

$$S_{\Sigma}(t)=\sum_{k=-\infty}^{\infty} S_{T_M}(t-kT_M)=\frac{1}{T_M}\sum_{k=-\infty}^{\infty} G_{T_M}\left(\frac{k}{T_M}\right)e^{j2\pi\frac{k}{T_M}t}. \qquad (1.25)$$

It is assumed that the function $S_{T_M}(t)$ is identically equal to zero outside the interval $(0, T_M)$. The latter is stipulated in order for the function in question to approximate radar signals, but is not a bound which flows from the exposition above. The other quantities, such as G, Ω, Ψ, etc., referring to the same repetition period, are also given the subscript T_M, so that

$$G_{T_M}(f) \leftrightarrow S_{T_M}(t). \qquad (1.26)$$

The function (1.25) has a discrete frequency spectrum. However, by using the concept of delta-function $\delta(\chi)$ we can arrive at a generalized representation with which the spectral density function also includes separate spectral lines, including the constant component [8]

$$\sum_{k=-\infty}^{\infty} S_{T_M}(t-kT_M) \leftrightarrow \frac{1}{T_M}\sum_{k=-\infty}^{\infty} G_{T_M}\left(\frac{k}{T_M}\right)\delta\left(f-\frac{k}{T_M}\right). \qquad (1.27)$$

Operations with a delta-function based on these formal relationships

$$\delta(t-t_0) \leftrightarrow e^{-j2\pi f t_0}, \qquad (1.28)$$

$$e^{j2\pi f_0 t} \leftrightarrow \delta(f-f_0),$$

$$\int_{-\infty}^{\infty} f(x)\delta(y-x)\,dx = f(y), \qquad (1.29)$$

$$\sum_{k=-\infty}^{\infty} \delta(t-kT) \doteq \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{j2\pi\frac{k}{T}t} \qquad (1.30)$$

$$T\sum_{k=-\infty}^{\infty} e^{j2\pi fkT} = \sum_{k=-\infty}^{\infty} \delta\left(f-\frac{k}{T}\right), \tag{1.31}$$

can be carried out sufficiently rigorously. In order to do this it is necessary only to carry out the entire operation with the function with the function $\frac{1}{\tau}\exp\left(-\frac{\pi t^2}{\tau^2}\right)$, or with another impulse function having unit area, then passing to the limit $\tau \to 0$.

The complex envelope S(t) of a reflected radar signal usually represents the product of a periodic function $S_\Sigma(t)$ and a cutoff function 1(t;T), where T is the effective duration of the cutoff function

$$T = \int_{-\infty}^{\infty} 1^2(t, T)\,dt. \tag{1.32}$$

We shall be using below the rectangular cutoff function

$$1_n(t, T) = \left\{\begin{array}{ll} 1 & \text{for } |t| < 0{,}5T \\ & \text{for } |t| > 0{,}5T \end{array}\right\} \leftrightarrow T\frac{\sin \pi Tf}{\pi Tf} \tag{1.33}$$

and the bell-shaped cutoff function

$$1_e(t, T) = \exp\left(-\frac{\pi t^2}{2T^2}\right) \leftrightarrow \sqrt{2}\,T\exp(-2\pi T^2 f^2). \tag{1.34}$$

The frequency spectrum of the product is equal to the convolution of the frequency spectra of the factors, so that

$$S(t) = 1_n(t, T)\sum_{k=-\infty}^{\infty} S_{T\,\text{M}}(t-kT_\text{M}) \leftrightarrow \frac{T}{T_\text{M}}\sum_{k=-\infty}^{\infty} G_{T\,\text{M}}\left(\frac{k}{T_\text{M}}\right)\times \times\frac{\sin \pi T\left(f-\frac{k}{T_\text{M}}\right)}{\pi T\left(f-\frac{k}{T_\text{M}}\right)} \tag{1.35}$$

and

$$S(t) = 1_e(t, T) \sum_{k=-\infty}^{\infty} S_{T_M}(t - kT_M) \leftrightarrow \frac{\sqrt{2}T}{T_M} \sum_{k=-\infty}^{\infty} G_{T_M}\left(\frac{k}{T_M}\right) \times$$
$$\times \exp\left[-2\pi T^2 \left(f - \frac{k}{T_M}\right)^2\right]. \tag{1.36}$$

The following general rule can be formulated. When multiplying a periodic function and a cutoff function, the Fourier transform becomes solid rather than remaining a line spectrum. Furthermore, each spectral line "spreads" into a spectral band which has a form similar to the Fourier transform of the cutoff function.

Adjacent spectral bands of the frequency spectra (1.35) and (1.36) practically do not overlap. For example, the adjacent frequency bands

$\exp\left[-2\pi T^2 \left(f - \frac{n}{T_M}\right)^2\right]$ and $\exp\left[-2\pi T^2\left(f - \frac{n+1}{T_M}\right)^2\right]$, even with a

relatively small ratio between the signal duration T and the modulation period T_M totaling 2, intersect at the 0.002 level. We shall be using this fact later in various calculations. As an illustration, we present here the calculation of the correlation function of the modulation of the signal (1.36). Assuming that adjacent spectral bands do not overlap, the power spectrum of the modulation can be assumed to be

$$\Omega(f) = 2\left(\frac{T}{T_M}\right)^2 \sum_{k=-\infty}^{\infty} \Omega_{T_M}\left(\frac{k}{T_M}\right) \exp\left[-4\pi T^2 \left(f - \frac{k}{T_M}\right)^2\right]. \tag{1.37}$$

The latter expression represents the convolution of the functions

$$\frac{1}{T_M}\left(\frac{T}{T_M}\right) \sum_{k=-\infty}^{\infty} \Omega\left(\frac{k}{T_M}\right) \delta\left(f - \frac{k}{T_M}\right) \tag{1.38}$$

and

$$2T \exp[-4\pi T^2 f^2],$$

the Fourier transforms of which will accordingly be

$$\sum_{k=-\infty}^{\infty} \frac{T}{T_{\text{м}}} Q^2_{T\,\text{м}} \Psi_{T\,\text{м}}(\tau - kT_{\text{м}})$$

and (1.39)

$$\exp\left(-\frac{\pi\tau^2}{4T^2}\right),$$

where

$$Q^2_{T\,\text{м}} = \frac{1}{2} \int_{-\infty}^{\infty} |S_{T\,\text{м}}(t)|^2\, dt. \tag{1.40}$$

and

$$\Psi_{T\,\text{м}}(\tau) = \frac{1}{2Q^2_{T\,\text{м}}} \int_{-\infty}^{\infty} S_{T\,\text{м}}(t)\, \overline{S_{T\,\text{м}}(t-\tau)}\, dt. \tag{1.41}$$

Therefore

$$\Psi(\tau) = \exp\left(-\frac{\pi\tau^2}{4T^2}\right) \sum_{k=-\infty}^{\infty} \Psi_{T\,\text{м}}(\tau - kT_{\text{м}}). \tag{1.42}$$

Analogously, for the signal (1.35) the correlation function of the modulation is

$$\Psi(\tau) = \begin{cases} \left(1 - \frac{|\tau|}{T}\right) \sum\limits_{k=-\infty}^{\infty} \Psi_{T\,\text{м}}(\tau - kT_{\text{м}}) & \text{for} \quad |\tau| \leqslant T, \\ 0 & \text{for} \quad |\tau| \geqslant T. \end{cases} \tag{1.43}$$

1.2. Some Definitions and Formulas for Stationary Random Functions

Of major importance for the theory of stationary random processes is Khinchin's formula [51], according to which the spectral intensity function $\nu(f)$ of a random process $n(t)$ and its correlation function

$$\psi_n(\tau) = \frac{1}{\sigma^2} \langle n(t) n(t-\tau) \rangle, \qquad (1.44)$$

are connected by the pair of Fourier transforms

$$\nu(f) \leftrightarrow \sigma^2 \psi_n(\tau), \qquad (1.45)$$

where

$$\sigma^2 = \langle n^2(t) \rangle, \qquad (1.46)$$

with the angle brackets $\langle \quad \rangle$ here and below designating statistical averaging. It is assumed for simplicity that the mathematical expectation of the process n(t) is zero.

The reception interference examined in this study is represented by a stationary normal process, usually called a random process. We shall distinguish between two types of random processes: high frequency n(t) and lower frequency N(t). The spectral intensity of a low frequency random process N(t) is the low frequency function $\mathcal{N}^0$ (f) which is almost wholly concentrated in the vicinity of the point f=0. Furthermore, the origin point on the f axis is defined by a formula analogous to (1.19).

The spectral intensity ν(f) of a high frequency random process, like the power spectrum of determinate signals, can be represented as

$$\nu(f) = \frac{1}{2} \mathcal{N}^0(f - f_0) + \frac{1}{2} \mathcal{N}^0(-f - f_0), \qquad (1.47)$$

where $\mathcal{N}^0$ (f) is a low frequency function which is practically non-zero on an interval whose length is significantly smaller than the center frequency f_0.

Substituting (1.47) in Khinchin's formula (1.45), we find

$$\langle n(t) n(t-\tau) \rangle = \left[\int_{-\infty}^{\infty} \mathcal{N}^0(f) \cos(2\pi f \tau)\, df \right] \cos 2\pi f_0 \tau - \\ - \left[\int_{-\infty}^{\infty} \mathcal{N}^0(f) \sin(2\pi f \tau)\, df \right] \sin 2\pi f_0 \tau. \qquad (1.48)$$

A random process can be represented analytically in different ways. We recall that the spectral intensity $\nu(f)$ or its Fourier transform $\langle n(t)n(t-\tau)\rangle$ characterizes a random process exhaustively. Therefore, the criterion for the correctness of a selected representation can be the coincidence of the spectral intensity $\nu(f)$ or average products $\langle n(t)n(t-\tau)\rangle$ of the analytical representation and the real process.

A high frequency random process is often represented [2, 18] as the sum of sinusoidal oscillations which are amplitude-modulated by low frequency random processes $N_c(t)$ and $N_s(t)$

$$n(t)=N_c(t)\cos(2\pi f_0 t+\varphi)+N_s(t)\sin(2\pi f_0 t+\varphi) \tag{1.49}$$

or

$$n(t)=\mathrm{Re}\{N(t)e^{j(2\pi f_0 t+\varphi)}\}, \tag{1.50}$$

where ϕ -- arbitrary initial phase, and

$$N(t)=N_c(t)-jN_s(t). \tag{1.51}$$

If the high frequency process n(t) has spectral intensity $\nu(f)$ assigned by means of (1.47), the processes $N_c(t)$ and $N_s(t)$ must be such that

$$\langle N_c(t)N_c(t-\tau)\rangle=\langle N_s(t)N_s(t-\tau)\rangle=\int_{-\infty}^{\infty}\mathcal{N}(f)\cos(2\pi f\tau)\,df, \tag{1.52}$$

$$\begin{aligned}\langle N_c(t)N_s(t-\tau)\rangle&=-\langle N_c(t-\tau)N_s(t)\rangle=\\&=\int_{-\infty}^{\infty}\mathcal{N}(f)\sin(2\pi f\tau)\,df.\end{aligned} \tag{1.53}$$

The average product $\langle n(t)n(t-\tau)\rangle$ is the same as (1.48).

In accordance with the real conditions encountered in practical applications, we can then assume that the function $\mathcal{N}(f)$ is even

$$\mathcal{N}(-f)=\mathcal{N}(f). \tag{1.54}$$

Then

$$\langle N_c(t)\,N_c(t-\tau)\rangle=\langle N_s(t)\,N_s(t-\tau)\rangle=\int_{-\infty}^{\infty}\mathcal{N}(f)\,e^{j2\pi f\tau}df, \tag{1.55}$$

$$\langle N_c(t)\,N_s(t-\tau)\rangle=\langle N_c(t-\tau)\,N_s(t)\rangle=0, \tag{1.56}$$

$$\langle N(t)\,N(t-\tau)\rangle=2\,\langle N_c(t)\,N_c(t-\tau)\rangle \tag{1.57}$$

and

$$\langle n(t)\,n(t-\tau)\rangle=\langle N_c(t)\,N_c(t-\tau)\rangle\cos 2\pi f_0\tau. \tag{1.58}$$

In this case, according to (1.56), the random processes $N_c(t)$ and $N_s(t)$ are statistically independent.

It is also possible to represent a high frequency random process n(t) as

$$n(t)=N_{c\varkappa}(t)\cos[2\pi f_0t+\varkappa(t)+\varphi]+N_{s\varkappa}(t)\times \\ \times\sin[2\pi f_0t+\varkappa(t)+\varphi]. \tag{1.59}$$

The determinate function $\kappa(t)$ defines the angle modulation principle. If the low frequency random processes $N_{CK}(t)$ and $N_{SK}(t)$ are such that

$$\langle N_{c\varkappa}(t)\,N_{c\varkappa}(t-\tau)\rangle=\langle N_{s\varkappa}(t)\,N_{s\varkappa}(t-\tau)\rangle= \\ =\cos[\varkappa(t)-\varkappa(t-\tau)]\int_{-\infty}^{\infty}\mathcal{N}(f)\,e^{j2\pi f\tau}df, \tag{1.60}$$

$$\langle N_{c\varkappa}(t-\tau)\,N_{s\varkappa}(t)\rangle=-\langle N_{c\varkappa}(t)\,N_{s\varkappa}(t-\tau)\rangle= \\ =\sin[\varkappa(t)-\varkappa(t-\tau)]\int_{-\infty}^{\infty}\mathcal{N}(f)\,e^{j2\pi f\tau}df, \tag{1.61}$$

the average product is

$$\langle n(t)\,n(t-\tau)\rangle=\cos 2\pi f_0\tau\int_{-\infty}^{\infty}\mathcal{N}(f)\,e^{j2\pi f\tau}df, \tag{1.62}$$

i.e., the same as (1.58)

The derivative N'(t) of an arbitrary differentiable random process is also a random process [22]. The statistical characteristics of the function N'(t) can be found from the statistical characteristics of the original function N(t). By measuring the order of differentiation and statistical averaging (integration), we obtain

$$\langle N'(t)\,N'(t-\tau)\rangle = -\frac{d^2}{d\tau^2}\langle N(t)\,N(t-\tau)\rangle, \tag{1.63}$$

$$\langle N'(t)\,N(t)\rangle = 0. \tag{1.64}$$

The spectral intensity function $\mathcal{N}$ (f) of the random process N(t) is often wider than the spectral lines of other temporal processes examined in conjunction with random ones. The spectral intensity $\mathcal{N}$(f) is approximately constant within the frequency domain employed in the present problem. The rate at which the function $\mathcal{N}$(f) drops off outside the working frequency region is of no significance from the practical point of view. Therefore, a wideband random process N(t) can be approximated by a hypothetical process -- uncorrelated noise (white noise) -- characterized by the following relationships:

$$\mathcal{N}(f) = N_0 = \text{const}, \tag{1.65}$$

$$\langle N(t_1)\,N(t_2)\rangle = N_0\delta(t_1 - t_2). \tag{1.66}$$

The spectral intensity ν(f) of a corresponding high frequency process n(t) consisting of two sidebands $\frac{1}{2}\mathcal{N}(f-f_0)$ and $\frac{1}{2}\mathcal{N}(-f-f_0)$, will be

$$\nu(f) = n_0 = \frac{N_0}{2} \tag{1.67}$$

and

$$\langle n(t_1)\,n(t_2)\rangle = \frac{N_0}{2}\,\delta(t_1 - t_2). \tag{1.68}$$

It should be emphasized that white noise cannot be realized, and that it represents only a convenient mathematical idealization. For example, a real normal process N(t) with correlation time τ_K interacting on a system with a time constant significantly larger than τ_K behaves like white noise with spectral intensity

$$N_0 = \int_{-\infty}^{\infty} \langle N(t) N(t-\tau) \rangle \, d\tau.$$

We note that we are employing representation of frequency spectra along the entire frequency axis from $f = -\infty$ to $f=\infty$, while only positive frequencies have any physical meaning. Therefore, the spectral intensity which figures in our formulas, such as $\nu(f)$ for the random process n(t), represents a quantity half as large as the real spectral intensity, which means the average noise power per Hertz of bandwidth.

The following relationships hold for random processes which can be approximated by white noise:

$$1.\quad \left\langle \iint_{-\infty}^{\infty} N(t_1) N(t_2) \varphi(t_1, t_2) \, dt_1 \, dt_2 \right\rangle = N_0 \int_{-\infty}^{\infty} \varphi(t_1, t_1) \, dt_1, \tag{1.69}$$

$$2.\quad \left\langle \iiiint_{-\infty}^{\infty} N_c(t_1) N_c(t_2) N_s(t_3) N_s(t_4) \varphi(t_1, t_2, t_3, t_4) \, dt_1 dt_2 dt_3 dt_4 \right\rangle =$$

$$= N_0^2 \iint_{-\infty}^{\infty} \varphi(t_1, t_1, t_2, t_2) \, dt_1 \, dt_2, \tag{1.70}$$

$$3.\quad \left\langle \iiiint_{-\infty}^{\infty} N(t_1) N(t_2) N(t_3) N(t_4) \varphi(t_1, t_2, t_3, t_4) \, dt_1 \, dt_2 \, dt_3 dt_4 \right\rangle =$$

$$= N_0^2 \Big[\iint_{-\infty}^{\infty} \varphi(t_1, t_1, t_2, t_2) \, dt_1 \, dt_2 + \iint_{-\infty}^{\infty} \varphi(t_1, t_2, t_1, t_2) \, dt_1 \, dt_2 + \tag{1.71}$$

$$+ \iint_{-\infty}^{\infty} \varphi(t_1, t_2, t_2, t_1) \, dt_1 \, dt_2 \Big].$$

where $\phi(t_1,t_2)$ and $\phi(t_1,t_2,t_3,t_4)$ are arbitrary numerical functions of the variable t_1, t_2 and t_1, t_2,t_3,t_4. Expressions (1.69) and (1.70) are obtained by direct application of (1.66) and (1.56); the derivation of formula (1.71) is provided in [40].

A random process can often be represented as the convolution of white noise with some determinate function. Therefore, formulas (1.69)-(1.71), in part, can be used to compute different moments of random processes.

Let us assume, for example, that the random process $n_1(t,\alpha)$ is the result of convolution of white noise n(t) with a determinate function of time t and the parameter α-$s(t,\alpha)$

$$n_1(t, \alpha) = \int_{-\infty}^{\infty} n(z) s(t-z, \alpha) dz: \tag{1.72}$$

Then, on the basis of (1.69),

$$\langle n_1(t_1, \alpha_1) n_1(t_2, \alpha_2)\rangle = n_0 \int_{-\infty}^{\infty} s(t+t_1, \alpha_1) s(t+t_2, \alpha_2) dt. \tag{1.73}$$

If

$$N_1(t, \alpha) = \mathrm{Re}\left\{ \int_{-\infty}^{\infty} N(z) S(t-z, \alpha) dz \right\}, \tag{1.74}$$

then

$$\langle N_1(t_1, \alpha_1) N_1(t_2, \alpha_2)\rangle = \\ = N_0 \mathrm{Re}\left\{ \int_{-\infty}^{\infty} S(t+t_1, \alpha_1) \overline{S(t+t_2, \alpha_2)} dt \right\}. \tag{1.75}$$

As a second example, let us find the mathematical expectation and dispersion of the function

$$W(\tau)=\frac{1}{T}\int_{-\infty}^{\infty} n_1(t)\,1(t,T)\,n_1(t-\tau)\,1(t-\tau,T)\,dt, \tag{1.76}$$

which is a generalization of a short-time correlation function [57]. Process $n_1(t)$ is defined via (1.72), assuming that $s(t,\alpha)=s(t)$. Using (1.69) and (1.71), we obtain

$$\langle W(\tau)\rangle = n_0 Q^2 \psi_1(\tau)\,\psi(\tau), \tag{1.77}$$

$$\langle [W(\tau)-\langle W(\tau)\rangle]^2\rangle =$$
$$=\frac{n_0^2 Q^4}{T}\int_{-\infty}^{\infty} \psi_1(\tau;\; x)\,[\psi^2(x)+\psi(x+\tau)\,\psi(x-\tau)]\,dx, \tag{1.78}$$

where

$$\psi_1(\tau)=\frac{1}{T}\int_{-\infty}^{\infty} 1(t,\;T)\,1(t-\tau,\;T)\,dt,$$

$$\psi_1(\tau,\;x)=$$
$$=\frac{1}{T}\int_{-\infty}^{\infty} 1(t,\;T)\,1(t-\tau,\;T)\,1(t-x,\;T)\,1(1-\tau-x,\;T)\,dt,$$

and $\psi(\tau)$ and Q^2 are the correlation function and total energy of the process $s(t)$.

In the case of a bell-shaped cutoff function

$$\psi_1(\tau)=\exp\left(-\frac{\pi\tau^2}{4T^2}\right),$$

$$\psi_1(\tau;\;x)=\frac{1}{\sqrt{2}}\exp\left(-\frac{\pi\tau^2}{2T^2}\right)\exp\left(-\frac{\pi x^2}{2T^2}\right).$$

We shall be representing various electrical oscillations which are certain functions of time as points or vectors in multidimensional space. The possibility of this representation is substantiated in

[18, 8, 63], and refined in a number of subsequent studies, which are reviewed in [43]. Accordingly, we shall represent the interference n(t) as the point $n=(n_1, n_2 \ldots, n_m)$, and we shall write the density of the probability p(n) that random interference on the interval $t_1 \leq t \leq t_2$ will take on a value of n (interference probability density functional) as [18, 8, 63].

$$p(n) = k \exp\left[-\frac{1}{N_0}\int_{t_1}^{t_2} n^2(t)\,dt\right]. \tag{1.79}$$

Expression (1.79) is derived on the assumption that the spectral intensity of random interference

$$\nu(f) = \begin{cases} \frac{N_0}{2} & \text{for } |f| < f_m, \\ 0 & \text{for } |f| > f_m. \end{cases}$$

Furthermore, the coefficient k in (1.79) is

$$k = \left(\frac{1}{2\pi N_0 f_m}\right)^{\frac{m}{2}};\ m = 2f_m(t_2 - t_1).$$

The boundary frequency f_m can be as high as desired. Therefore, expression (1.79) is used, in part, for a white noise approximation of interference. It must only be kept in mind that as $f_m \to \infty$ the coefficient $k \to \infty$, and there exists no finite limit for the function defined by formula (1.79). However, since the coefficient k is independent of the realization of the interference n(t), the ratio of the probability density functionals for two interference realizations, say $n^*(t)$ and $n^{**}(t)$, has a finite limit and physical meaning. The ratio of the probability density functionals

$\frac{p(n^*)}{p(n^{**})}$ indicates by how much the realization $n^*(t)$ is more or less probable than the realization $n^{**}(t)$. Thus, the probability density functional of

white noise is defined to within the coefficient k. This does not result in misunderstandings when using the concept of "white noise", since radar observation theory does not actually examine the interference probability density functionals, but rather their ratios.

Chapter 9 investigates the case in which the intensity of the interference N_0 is an unknown random quantity distributed as $p_1(N_0)$. In this case expression (1.79) for the interference probability density functional is transformed to the following:

$$p(n)=\left(\frac{1}{2\pi f_m}\right)^{\frac{m}{2}}\int_0^{\infty}\frac{p_1(N_0)}{N_0^{\frac{m}{2}}}\exp\left[-\frac{1}{N_0}\int_{t_1}^{t_2}n^2(t)\,dt\right]dN_0. \qquad (1.80)$$

1.3. Combined Transmission of Signal and Interference Through Receiver

We shall assume that the receiver consists of a linear radio frequency amplifier (incorporating in the general case an HF amplifier, a converter and an i.f. amplifier), an inertialess detector and a linear video amplifier, tuned to the frequency of the received signal. Input to the receiver is the oscillation

$$x(t)=n(t)+s(t-\tau_0), \qquad (1.81)$$

consisting of random interference, which can be approximated by white noise with spectral intensity $N_0/2$, and the valid signal

$$s(t-\tau_0)=S(t-\tau_0)\cos[2\pi f_0(t-\tau_0)+\varphi]. \qquad (1.82)$$

We recall that the real spectral intensity of the interference at the input of a receiver is N_0. We are examining the transmission of amplitude-modulated (or pulsed) signals. However, as will be shown later,

the findings can also be used for other types of radar signals.

We shall designate the frequency responses of the radio frequency amplifier and video amplifier a(f) and A(f), and their Fourier transforms, i.e., the impulse responses of amplifiers, h(t) and H(t). The frequency response of the radio frequency amplifier can be represented as

$$a(f)=a_{\Omega}(f-f_0)+a_{\Omega}(f+f_0), \tag{1.83}$$

where $a_{\Omega}(f)$ is a function which is practically non-zero in a comparatively (with f_0) narrow frequency band near f=0, such that

$$a_{\Omega}(-f)=\overline{a_{\Omega}(f)}.$$

Furthermore, introducing the notation

$$h_{\Omega}(t)\leftrightarrow a_{\Omega}(f), \tag{1.84}$$

we find

$$h(t)=\int_{-\infty}^{\infty} a(f)\,e^{j2\pi ft}\,df=2h_{\Omega}(t)\cos 2\pi f_0 t. \tag{1.85}$$

The function $2h_{\Omega}(t)$, determined by means of (1.84), is the envelope of the impulse response of the radio frequency amplifier.

We shall give the respective subscripts 1, 2 and 3 to the different voltages at the output of the radio frequency amplifier, the detector and the video amplifier.

The voltage $x_1(t)$ at the output of the radio frequency amplifier is equal to the sum of the signal voltage

$$s_1(t-\tau_0)=\int_{-\infty}^{\infty} s(t-\tau_0-z)\,h(z)\,dz=$$

$$=\cos[2\pi f_0(t-\tau_0)+\varphi]\int_{-\infty}^{\infty} S(z)\,h_\Omega(t-\tau_0-z)\,dz \qquad (1.86)$$

and the envelope

$$S_1(t-\tau_0)=\int_{-\infty}^{\infty} S(z)\,h_\Omega(t-\tau_0-z)\,dz \qquad (1.87)$$

[PAGES 24 AND 25 OF ORIGINAL TEXT OMITTED.]

2. Square-law detection of a strong signal

$$X_2(t) \approx S_1^2(t-\tau_0) + 2S_1(t-\tau_0)N_{c1}(t). \qquad (1.99)$$

3. A weak signal

$$X_2(t) \approx S_1^2(t-\tau_0) + N_{c1}^2(t) + N_{s1}^2(t). \qquad (1.100)$$

The latter expression reflects accurately enough the transmission of a weak signal through a square-law detector; it is extremely approximate for a linear detector. The accuracy of formula (1.100) for linear detection, however, is not important for us, since we are employing a square-law detector for weak signals (Chapters 6 and 8).

The case of linear detection of a strong signal is fundamental in the investigations below. We note that the output of a linear strong-signal detector is the same as the output of a synchronous detector. In both cases the phased component of the interference $N_{c1}(t)$ is retained at the output along with the signal envelope. The quadrature component $N_{s1}(t)$ is almost fully suppressed. With the matched frequency response (1.90) the valid component $S_1(t-\tau_0)$ and the correlation function of the interference $N_{c1}(t)$ at the output of a linear strong-signal detector have the same form as the modulation correlation function

$$S_1(t-\tau_0) = Q^2|\Psi(t-\tau_0)|, \qquad (1.101)$$

$$\langle N_{c1}(t_1)N_{c1}(t_2)\rangle = \frac{1}{2}N_0Q^2\Psi(t_1-t_2). \qquad (1.102)$$

Finally, the voltage $X_3(t)$ at the video amplifier output will be

$$X_3(t) = \int_{-\infty}^{\infty} X_2(z)H(t-z)\,dz. \qquad (1.103)$$

For linear detection of strong signals, the latter expression is transformed to

$$X_3(t)=\int_{-\infty}^{\infty}[S(z-\tau_0)+N_c(z)]H_\Sigma(t-z)\,dz, \tag{1.104}$$

where $H_\Sigma(t)$ is the generalized impulse response of the receiver

$$H_\Sigma(t)=\int_{-\infty}^{\infty}h_\Omega(z)H(t-z)\,dz, \tag{1.105}$$

which is the Fourier transform of the generalized frequency response $A_\Sigma(f)$

$$A_\Sigma(f)=a_\Omega(f)A(f). \tag{1.106}$$

Thus, in the case of linear detection of strong signals the video amplifier output voltage $X_3(t)$ is fully determined by the generalized response of the receiver $H_\Sigma(t)$ for $A_\Sigma(f)$, and it is irrelevant how this response is distributed among individual components of the receiver, especially between the i.f. amplifier and video amplifier.

Analogously, for square law detection of strong signals

$$X_3(t)=\iint_{-\infty}^{\infty}[S(z_1-\tau_0)S(z_2-\tau_0)+2S(z_1-\tau_0)N_c(z_2)]\times \times H_\Sigma(t-z_1,\ t-z_2)\,dz_1dz_2 \tag{1.107}$$

and for detection of weak signals

$$X_3(t)=\iint_{-\infty}^{\infty}[S(z_1-\tau_0)S(z_2-\tau_0)+N_c(z_1)N_c(z_2)+ +N_s(z_1)N_s(z_2)]H_\Sigma(t-z_1,\ t-z_2)\,dz_1dz_2, \tag{1.108}$$

where $H_\Sigma(t_1,t_2)$ -- generalized two-dimensional impulse response of receiver,

$$H_\Sigma(t_1, t_2) = \int_{-\infty}^{\infty} h_\Omega(t_1 - z)\, h_\Omega(t_2 - z)\, H(z)\, dz, \tag{1.109}$$

which is the Fourier transform of the generalized two dimensional frequency response $A_\Sigma(f_1, f_2)$

$$A_\Sigma(f_1, f_2) = a_\Omega(f_1)\, a_\Omega(f_2)\, A(f_1 + f_2), \tag{1.110}$$

i.e.,

$$H_\Sigma(t_1, t_2) = \iint_{-\infty}^{\infty} A_\Sigma(f_1, f_2) \exp[j2\pi(f_1 t_1 + f_2 t_2)]\, df_1 df_2. \tag{1.111}$$

Thus, in the case of square law detection of strong and weak signals the video amplifier output voltage $X_3(t)$ is determined by the generalized two-dimensional receiver response $H_\Sigma(t_1,t_2)$ or $A_\Sigma(f_1, f_2)$ and it is irrelevant how this response is distributed among the individual components of the receiver.

The case is also encountered, in Chapters 5 and 7, in which the envelope detector has as input the oscillation

$$x_1(t) = |S_1(t - \tau_0)| \cos[2\pi f_0 (t - \tau_0) + \varkappa(t - \tau_0) + \varphi] + n_1(t), \tag{1.112}$$

consisting of random interference and a signal with simultaneous amplitude and phase modulation. If we use the representation (1.59) for the random interference $n_1(t)$, the output of the strong-signal detector can be represented as

$$X_2(t) \approx |S_1(t - \tau_0)| + N_{c\kappa}(t). \tag{1.113}$$

The low frequency random process $N_{CK}(t)$ is defined by formula (1.60).

CHAPTER 2. Statistical Treatment of Radar Observation Process

2.1. Statistical Model of Radar Observation Process

The observation process performed by a radar operator consists of detecting signals reflected from targets, and estimating certain parameters of these signals, such as the parameters which determine the target coordinates.

The interference which unavoidably occurs in any radio link masks the valid signals and makes it impossible to use the received oscillation to establish with absolute certaintly that a valid signal is present and to estimate its parameters precisely. The occurrence of interference causes the observation results to be random, and makes it necessary to study these results by statistical methods.

Under actual operating conditions the decisions made by an operator when detecting signals and in estimating signal parameters are based on certain subjective criteria which depend upon the level of training and other properties of the operator. Subjective operator properties have some influence on radar observation performance indicators. However, the limiting capabilities and indicators of radar systems are determined primarily by the statistical nature of the problem. A statistical statement of the problem requires only that we free the radar observation process of subjective "baggage", assuming that the operator acts on the basis of clearly formulated decision making rules, and replacing the operator with a decision device. Replacing the operator with a machine -- a computing and decision device -- also corresponds to the current trend in the development of radar systems.

We shall represent the statistical model of the radar observation process to be examined below in the form of the diagram shown in Fig. 2.1. The main components of this diagram are the signal space Σ, the interference space N, the received oscillation space X, the receiver, the output space Y, the decision device and the decision space $\tilde{S}^*$.

The signal space Σ consists of the set of all possible valid reflected signals arriving at the input of the receiver. The set of valid signals can be defined in principle such that it includes the case of receiving several reflected signals simultaneously. However, we shall assume that it is possible for only one single target to be present in the space which the radar is examining. A brief discussion of the possibility of the simultaneous presence of a large number of targets (resolution problem) will be presented later.

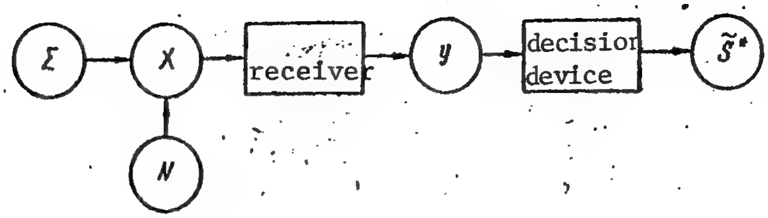

Fig. 2.1. Statistical model of radar observation process

The reflected signal is formed as the result of interaction between the radiated signal, the reflecting object and the environment. The reflected signal can be represented analytically as some known function (determined by the nature of the radiated signal) of time and a series of random parameters. The random parameters in the reflected signal to be estimated will be called useful parameters. The random parameters of the reflected signal which are not to be estimated and which are statistically independent with respect to the useful parameters, i.e., not containing information about the useful parameters, will be called parasitic parameters.

The reflected signal can thus be written as the function $\text{з}(t,s,u)$, where s and u are random vector quantities which represent respectively the set of useful and parasitic parameters. We note that the reflected signal can consist of the sum of the elementary signals $\text{з}=\text{з}_1+\ldots+\text{з}_n$), differing from one another in carrier frequency, modulation principle or by the fact that they represent the state of the electromagnetic field at different points in real space. It is assumed that the reflected signal $\text{з}(t,s,u)$ has a finite fixed duration of T sec. The signal space, besides the set of functions $\text{з}(t,s,u)$, also includes the null signal which denotes the absence of a target. We shall designate the event "target absent" by the symbol s_0 and consider that s_0 is one of the possible values of the random vector quantity s.

Each point of the signal space Σ is thus uniquely determined by the values taken on by the parameters s and u. It will be helpful below to introduce as well the concept of useful parameter space $\tilde{S}$, or the set of all possible values of the random parameter s.

As a rule, random parameters can take on arbitrary values within certain intervals. However, in most theoretical and experimental studies it is useful to idealize the signal space somewhat in order to determine the basic dependencies and regularities. It can be assumed, for example, that some of the parasitic parameters, or even all of the parasitic parameters, in a reflected signal are non-random given quantities. This idealization makes it possible to determine separately the influence of each of the parasitic parameters on the radar observation process.

Analogously, it can be assumed that only a single useful parameter is to be estimated, such as the delay time, and that the remaining useful parameters are fixed, i.e., given. In this case the scanning is done with respect to only one parameter or one coordinate, such as the range coordinate in some fixed direction. The latter assumption makes it possible to resolve a number of important problems in radar theory.

Finally, an idealization is employed in which the set of useful parameters s can take on only a finite number of fixed values s_1, s_2..., s_m, as well as the value s_0. Radar lines with a discrete set of useful parameters will be called simplified radar lines.

The present chapter imposes no limitations on the class of interference contained in the space N. It is assumed only that the distribution p(n) is defined for the random vector quantity n representing the realization of the interference from the class in question. Added together, the interference and the signal form the received oscillation x(t), which is represented by the point x in the space X. The system must provide a unique transformation of the received oscillation space X to the decision space $\tilde{S}^*$, which consists of the set of all possible decisions s* made during the radar observation process. The decision space has the same structure as the useful parameter space $\tilde{S}$.

Using as guidance the consideration of convenience in practical construction of the system, the latter is assumed to consist of two parts: a receiving device and a decision device. Accordingly, the operation of transforming the received oscillation into a decision is split into two stages. The first stage involves transforming the received oscillation into the output of the receiver; the second stage consists of transforming the output into a decision.

2.2. Statement of Problem of Optimal System Definition

An optimal system is one which provides the decisions or estimates which are best in terms of a selected optimality criterion. The optimality criterion is selected so that it corresponds sufficiently well to the nature of the problem being solved. This rule does not define the criterion in only a single way, and leaves open the possibility of substantial arbitrariness in selecting it. The extent to which the selected criterion can serve as the basis for analytical solution of the problem should also be kept in mind.

An estimate is a random quantity, and its quality can be determined only over a large number of trials. For this reason, practically all optimality criteria can be reduced to ensuring the minimum or maximum mathematical expectation of some function of the signal and decision (estimate). The general optimality criterion for the radar observation process can thus be written as [28]

$$\rho = \int\limits_{\tilde{S}^*} \int\limits_{\tilde{S}} r(s^*, s)\, \tilde{p}(s)\, \tilde{p}(s^*/s)\, ds^* ds = \text{extremum}, \tag{2.1}$$

where s* is the estimate of the random quantity s, i.e., the decision made as the result of making the radar observation. The set of all possible decisions $\tilde{S}^*$, which includes integration with respect to s*, consists of the interval S* corresponding to the working range of variation of the useful signal parameters, and of the point s_0^* ("target absent" decision) outside the interval S*. For convenience in writing criterion (2.1) the decision set $\tilde{S}^*$ is augmented by the vicinity of the point s_0^*. The useful parameter set $\tilde{S}$ has the same structure as the set $\tilde{S}^*$, i.e., it includes the working interval S of variation of all of the useful parameters s and the point s_0 together with its vicinity.

r(s*,s) is the value function, which determines the relative value or significance of the combined realization of a signal with parameters s and the decision s*. In all cases here and below in which no misunderstanding can arise, the probability densities and probabilities of different quantities are denoted by the same letters: p -- probability density, P -- probability. Conditional probabilities are designated by the same symbols as unconditional ones, and differ only by the presence of a slash in the argument, to the right of which is indicated the conditional quantity or event. For example, p(s*/s) -- conditional probability density for the quantity s* given realization of the

quantity s (signal with useful parameters s). The character ~ over the symbol p means that the probability density function includes the discrete point s_0^* or s_0. Thus

$$\tilde{p}(s)=\begin{cases}p(s), & s\in \dot{S},\\ P(s_0)\,\delta(s-s_0), & s\bar{\in} S,\end{cases}$$
$$\tilde{p}(s^*/s)=\begin{cases}p(s^*/s), & s^*\in S^*,\\ P(s_0^*/s)\,\delta(s^*-s_0^*), & s^*\bar{\in} S^*.\end{cases} \tag{2.2}$$

The symbol $\in$ indicates "membership", while the symbol $\bar{\in}$ indicates non-membership. The function $\tilde{p}(s)$ determines the a priori probability of different outcomes.

Simplified lines can be viewed as a special case of real lines, for which the set of useful parameters s can take on only one of the m+1 fixed values s_0, $s_1 \ldots,$ s_m. When using simplified lines it should be assumed that

$$p(s)=\sum_{i=1}^{m} P(s_i)\,\delta(s-s_i),$$
$$p(s^*/s)=\sum_{i=1}^{m} P(s_i^*/s)\,\delta(s^*-s_i^*). \tag{2.3}$$

Accordingly, the general optimality criterion (2.1) for simplified lines becomes

$$\rho=\sum_{i=0}^{m}\sum_{j=0}^{m} r(s_j^*,\ s_i)\,P(s_i)\,P(s_j^*/s_i)=\text{extremum}. \tag{2.4}$$

The problem of defining an optimal system consists of determining the correspondence (called the system operator) between the received oscillation x and the decision s*

$$s^* = A(x), \tag{2.5}$$

which would satisfy the optimality condition -- condition (2.1) in the general case. Determination of the optimal operator actually means determining optimal processing of the received oscillations in the receiver and an optimal decision rule in the decision device. A practically important extension of the problem of defining an optimal system is the calculation of the performance indicators of the system. The performance indicators of an optimal system are called the limiting or potential indicators, since they characterize the theoretically limiting capabilities of the system.

The solution of the problem can be found in general form for real lines. Then the decision for simplified lines is obtained by applying formulas (2.3). However, it is more convenient to begin our examination with simplified lines, since this makes it possible to retain the connection with the customary representations which have been developed primarily for the two-alternative signal-no signal situation [6, 15, 61, 63, etc.].

Criteria (2.1) and (2.4) are Bayesian if the function $r(s^*,s)$ is assigned in advance and its form is independent of the decision making rule. Of all known practical criteria, only the criterion of minimum information loss, for which

$$r(s^*, s) = -\log p(s/s^*),$$

does not belong to the class of Bayesian criteria [28]. In investigating the radar observation process, the minimum information loss criterion provides essentially no advantage, but it makes problem solving much harder. Therefore, it is assumed below that the function $r(s^*,s)$ is independent of the manner in which the estimate s^* is obtained.

2.3. Simplified Lines

The oscillation x is input to the receiver. After appropriate processing of this oscillation it is necessary to make some decision s^*_j $(j=0,1,\ldots,m)$.

In the final analysis, without separating the functions of the receiving and decision devices, the decision is based on the fact that the received oscillation space X is divided into $m+1$ regions: X_0, $X_1 \ldots$, X_m.

Depending upon the region into which falls the point x, representing the received oscillation x(t), a corresponding decision s_j^* is made. This means that correspondence (2.5) takes on the appearance

$$s^* = s_j^* \text{ for } x \in X_j \; (j = 0, 1, \ldots, m) \tag{2.6}$$

and it is necessary only to define the algorithm for dividing the received oscillations space X into different decision regions.

The probability $P(s_j^*/s_i)$ that for the signal s_i arriving at the input of the receiving device together with interference the decision s_j^* will be made, which is the same as the probability that the oscillation x will fall in the region X_j when a signal with parameters s_i is present,

$$P(s_j^*/s_i) = \int_{X_j} p(x/s_i)\,dx. \tag{2.7}$$

The conditional distribution function p(x/s) of the quantity x, viewed as a function of s with a fixed value of x, is called the likelihood function. For additive interference the analytical expression of the function p(x/s) is determined directly by the given noise distribution p(n).

The quantities in (2.7) with the same subscripts for the signals and decisions define the conditional correct decision probabilities. The quantities in (2.7) with different signal and decision subscripts define the conditional probabilities of various types of decision making errors.

An error in which the decision s_i^* (i=1,2,..., m) is made that one of the non-null signals is present when there is no signal at the input, is called a false alarm. The false alarm probability F is

$$F = \sum_{i=1}^{m} P(s_i^* | s_0) = 1 - P(s_0^* | s_0). \tag{2.8}$$

An error in which the decision s_0^* -- "no target" is made when one of the non-null signals is present at the input, is called a missed signal.

We shall assume that the value function r is defined such that the equation ρ = max is the optimality criterion. Before defining the algorithm in which we are interested for dividing the received oscillation space, we shall formulate the following lemma.

If $\phi(x,0)$, $\phi(x,1),\ldots,$ $\phi(x,m)$ -- is a set of functions defined in the space X such that the equality $\phi(x,i)=\phi(x,j)$ for $i\neq j$ can occur only on the set of points of measure zero, and X_0, $X_1,\ldots,$ X_m are non-intersecting regions filling the space $X\left(\sum_{i=0}^{m} X_i = X\right)$, then

$$\sum_{i=0}^{m} \int_{X_i} \varphi(x, i)\, dx = \max \tag{2.9}$$

only when the regions X_i $(i=0,1,\ldots, m)$ are defined through the system of inequalities

$$\varphi(x, i) \geqslant \varphi(x, j), \; (j=0, 1, \ldots, m, \; j \neq i). \tag{2.10}$$

Let us compose the function

$$\Phi(x) = \sup[\varphi(x, 0), \varphi(x, 1), \ldots, \varphi(x, m)], \tag{2.11}$$

where the symbol sup (supremum) denotes the largest of the values contained in the square brackets. Then

$$\int_X \Phi(x)\, dx \geqslant \sum_{i=0}^{m} \int_{X_i} \varphi(x, i)\, dx. \tag{2.12}$$

The equals sign occurs in the latter expression only if for all i

$$\int_{X_i} \Phi(x)\,dx = \int_{X_i} \varphi(x,\ i)\,dx, \tag{2.13}$$

which leads directly to (2.10).

The stipulated condition that the measure of the set of points at which $\phi(x,i)=\phi(x,j)$ is zero can be omitted. Then the $\geq$ and $>$ signs in system of inequalities (2.10) must be combined. For example, for all $j>i$ we write the sign $\geq$, and for all $j<i$ we write the sign $>$. Keeping this remark in mind, we can later not concern ourselves with weather or not the aforementioned condition is satisfied.

We now transform expression (2.4) as follows:

$$\rho = \sum_{j=0}^{m} \int_{X_j} \sum_{l=0}^{m} r(s_j^*,\ s_l)\,P(s_l)\,p(x\mid s_l)\,dx = \max. \tag{2.14}$$

Based on the proof above, we conclude that condition (2.14) of the maximum of the quantity ρ is satisfied if the system of inequalities

$$\sum_{l=0}^{m} r(s_j^*,\ s_l)\,P(s_l)\,p(x\mid s_l) \geqslant \sum_{l=1}^{m} r(s_k^*,\ s_l)\,P(s_l)\,p(x\mid s_l) \quad (k=0,\ 1,\ldots,\ m,\ k\neq j). \tag{2.15}$$

is used as the definition of the region X_j $(j=0,1,\ldots,\ m)$.

The problem is solved completely analogously when the value function r is defined such that the optimality criterion consists of minimizing the quantity ρ. In this case it is necessary only to introduce the function inf (infinum), which denotes the smallest value, instead of the function sup. This makes it necessary to replace the $\geq$ sign in the final result of (2.15) with $\leq$.

This separation of the regions makes it possible to define, in general form for simplified lines (i.e., based on a general optimality criterion),

optimal processing of the received oscillations in the receiver and an optimal decision making rule for the decision device.

The boundary between the receiving and decision devices is somewhat arbitrary, and is determined by considerations of convenience in practical circuit implementation. It is possible to have an arbitrary unique variation in the output of the receiver with a corresponding variation in the decision making rule. It is also possible for various operations on the received oscillations to be related to either the receiving or the decision device. It is only necessary that the joint action of the receiving and decision devices ensure that system of inequalities (2.15) is satisfied when decisions are made. For example, when the oscillation x is received, the receiver might output the sequence of the quantities $P(s_i)\ p(x|s_i)$ for all values of i (the Woodward-Davis system). In this case the decision device multiplies the output by the corresponding coefficients of $r(s_j^*,s)$, performs the summation

$$\Sigma_j = \sum_{i=0}^{m} r(s_j^*,\ s_i)\, P(s_i)\, p(x\,|\,s_i) \tag{2.16}$$

and compares the values of the sums Σ_j with one another. The decision s_j^* is made that that signal s_j is present for which the quantity Σ_j is greatest. The final result remains the same if the operations of multiplication by $r(s_j^*,\ s_i)$ and summation are done by the receiving device or, conversely, if the receiver is relieved of multiplying by $P(s_i)$ by transferring that operation to the decision device.

In theoretical investigations it is convenient to use the concept of "likelihood coefficient" [63], which we shall designate by the symbol Λ and which is defined as

$$\Lambda(s) = \frac{p(x\,|\,s)}{p(x\,|\,s_0)}. \tag{2.17}$$

Like the likelihood function in the numerator of (2.17), the likelihood coefficient is a function of the signal parameters (or, in the general case, a function of the point in the signal space) for a fixed value of the received oscillation x.

Keeping (2.17) in mind, the following appearance can be given the system of inequalities which defines the division of the received oscillation space.

Region of null signal X_0

$$\sum_{i=1}^{m}[r(s_j^*, s_i) - r(s_0^*, s_i)] P(s_i) \Lambda(s_i) + $$
$$+ P(s_0) r(s_j^*, s_0) < P(s_0) r(s_0^*, s_0) \quad (j = 1, 2, \ldots, m). \tag{2.18}$$

Region of arbitrary non-null signal X_j

$$\sum_{i=1}^{m}[r(s_j^*, s_i) - r(s_0^*, s_i)] P(s_i) \Lambda(s_i) +$$
$$+ P(s_0) r(s_j^*, s_0^*) \geqslant P(s_0) r(s_0^*, s_0), \tag{2.19}$$

$$\sum_{i=1}^{m}[r(s_j^*, s_i) - r(s_0^*, s_i)] P(s_i) \Lambda(s_i) + P(s_0) r(s_j^*, s_0) \geqslant$$
$$\geqslant \sum_{i=1}^{m}[r(s_k^*, s_i) - r(s_0^*, s_i)] P(s_i) \Lambda(s_i) + P(s_0) r(s_k^*, s_0)$$
$$(k = 1, 2, \ldots, m;\ k \neq j). \tag{2.20}$$

The optimal decision algorithm obtained, based on the general optimality criterion, is fully applicable if for all real operating conditions the value functions $r(s_j^*, s_i)$ and the a priori distribution P (s_i^*) are defined.

However, in most cases this definition is lacking. First of all, the so-called "a priori difficulty" [8] occurs, which consists of the absence of enough a priori data to define the function $\tilde{p}(s)$. Second, statistical theory assumes the value function r(s*,s) to be given, and does not deal with selecting it. The value function r(s*,s) can in principle be defined (albeit not uniquely) based on the specific tactical conditions under which the system is used. In this case the function r will be different for different systems and for different operating conditions of the same system. On the other hand, as the function r changes, the meaning and results of the decision may change significantly. If the objective is to reduce theory to analytical formulas for system performance indicators, it is necessary to investigate systems with a specifically assigned

value function, instead of operations with the function r in general form.

Different optimality criteria which are encountered in solving practical problems are usually special cases of criterion (2.4) and (2.1) with various assumptions regarding the functions $\tilde{p}(s)$ and $r(s^*,s)$.

We note that the minimax method employed in game theory [1,55,62] does not allow the "a priori difficulty" in radar to be overcome satisfactorily. According to this method, the observation process is viewed as a game between an observer and nature. The value function provides a quantitative measure of the result of each realization of the game. It is assumed that nature selects a strategy which is least favorable for the observer, i.e., the function $\tilde{p}(s)$. With this assumption, based on Bayesian criterion (2.1) or (2.4), the observer chooses his strategy (decision making rule). Under actual operating conditions, the a priori distribution may differ significantly from the least favorable distribution. Accordingly, the decision made by the observer will not be the best ones. In statistical radar theory, the minimax method should be viewed as one possible, rather arbitrary method for assigning the lacking a priori data required in order to use the Bayesian criterion.

In accordance with existing engineering practice, we limit ourselves in practical applications to the Neumann-Pearson criteria and an ideal observer.

The ideal observer criterion [18, 26, 62], which provides the maximum average probability of making a correct decision

$$\rho = \sum_{i=0}^{m} P(s_i) P(s_i^* \mid s_i) = \max, \tag{2.21}$$

is obtained from general optimality criterion (2.4), if we assume

$$r(s_j^*, s_i) = \delta_{ji}, \tag{2.22}$$

where δ_{ji} -- Kronecker's symbol

$$\delta_{ji} = \begin{cases} 1 & \text{for } j = i, \\ 0 & \text{for } j \neq i. \end{cases} \tag{2.23}$$

After assumption (2.22) is made, there still remains a fundamental difficulty which is associated with the ambiguity of the a priori distribution. The simplest way out of this difficulty is to assign an arbitrary, but sufficiently likely a priori distribution, such as a uniform distribution

$$\begin{aligned} P(s_i) &= \frac{\mu}{m} \; (i=1,\, 2, \ldots, m), \\ P(s_0) &= 1-\mu, \end{aligned} \tag{2.24}$$

where μ -- probability of presence of non-zero signal, which must also be assigned. Then the systems of inequalities (2.18) and (2.19), which determine the separation of the space X into different decision regions, take on these forms:

-- region of null signal X_0

$$\Lambda(s_i) < \frac{1-\mu}{\mu} m \; (i=1,\, 2, \ldots, m), \tag{2.25}$$

-- region of arbitrary non-null signal X_i (i=1,2,...,m)

$$\begin{aligned} \Lambda(s_i) &\geqslant \frac{1-\mu}{\mu} m, \\ \Lambda(s_i) &\geqslant \Lambda(s_j) \; (j=1,\, 2, \ldots, m;\; j \neq i). \end{aligned} \tag{2.26}$$

The decision rule based on the Neumann-Pearson criterion leads to a similar result. The Neumann-Pearson criterion is usually defined for the two-alternative situation. We shall introduce an analogous definition for the case of an arbitrary number of alternatives.

We shall assume that according to the Neumann-Pearson criterion it is necessary to maximize the average conditional probability of making a correct decision with respect to non-null signals $\frac{1}{m}\sum_{i=1}^{m} P(s_i^* \mid s_i)$ assuming that the false alarm probability is equal to some given quantity F. The formulated criterion can be written analytically as follows:

$$\rho = \Lambda_0 P(s_0^*|s_0) + \sum_{i=1}^{m} P(s_i^*|s_i) =$$

$$= \int_{X_0} \Lambda_0 p(x/s_0)\,dx + \sum_{i=1}^{m} \int_{X_i} p(x|s_i)\,dx = \max \tag{2.27}$$

assuming that

$$\int_{X_0} p(x|s_0)\,dx = 1 - F, \tag{2.28}$$

where Λ_0 -- an arbitrary constant coefficient.

According to the above, of all of the possible algorithms for dividing the received oscillation space X into regions X_0, $X_1, \ldots, X_m$, the greatest value of the sum in (2.27) occurs when the definition of the arbitrary region X_i of a non-null decision s_i^* consists of the system of inequalities

$$\begin{aligned} &\Lambda(s_i) \geqslant \Lambda_0, \\ &\Lambda(s_i) \geqslant \Lambda(s_j) \quad (j=1,\ 2, \ldots,\ m,\ j \neq i), \end{aligned} \tag{2.29}$$

and the region X_0 of the null decision s_0^* is determined through the system

$$\Lambda(s_i) < \Lambda_0 \quad (i=1,\ 2,\ \ldots,\ m). \tag{2.30}$$

As Λ_0 varies from 0 to ∞ the region X_0 varies from 0 to X, the false alarm probability accordingly varies from 1 to 0. We select the quantity Λ_0 such that (2.28) occurs, i.e., such that the probability that system (2.30) will be satisfied when the null signal s_0 is input is exactly equal to the assigned quantity 1-F. The coefficient Λ_0 thus defined will be called

the critical (or threshold) level of the likelihood coefficient.

Systems of inequalities (2.29) and (2.30), assuming that the coefficient Λ_0 figuring in them is the critical level of the likelihood coefficient, facilitate simultaneous satisfaction of equations (2.27) and (2.28), and provide the sought separation of received oscillation space X into regions X_0, $S_1, \ldots, X_m$. Dividing the space X into any other regions $X'_0, \ldots, X'_m$ makes the sum of (2.27) smaller, and consequently reduces the quantity $\sum_{i=1}^{m} P(s_i^* \mid s_i)$, since the first term $\Lambda_0 P(s_0^* | s_0)$ must remain unchanged.

The Neumann-Pearson criterion cited above is obtained from the general optimality criterion (2.4) if we introduce the following initial assumptions: condition (2.22) and a uniform distribution of a priori probabilities (2.24) with a probability of occurrence of a non-null signal of

$$\mu = \frac{m}{m+\Lambda_0}. \qquad (2.31)$$

Thus, from the practical viewpoint it is completely irrelevant whether assumptions (2.24), which are arbitrary to some extent, are assigned with respect to the a priori probabilities when the ideal observer criterion is used, or whether the Neumann-Pearson criterion is used, which is also based on an arbitrary false alarm probability. In both cases the division of the received oscillations space is determined by systems of inequalities of the form (2.20) and (2.30). We note that when there are specific data concerning the a priori distribution and substantiated considerations regarding the choice of value function, allowance must be made for them in defining the decision making rule.

2.4. Real Lines

We shall modify general optimality criterion (2.1) somewhat. We first establish the following correspondence:

$$\rho_\sigma = \int_{\sigma^*} f(s^*, s)\, p(s^*/s)\, ds^* = \int_{X_\sigma} f(s^*, s)\, p(x/s)\, dx, \qquad (2.32)$$

where f(s*,s) is an arbitrary function of s* and s, X_σ is the region in

space X which is uniquely mapped onto the region σ^* of space S^* by using the operator of system (2.5). In order to prove (2.32) we partition the region σ^* into elementary regions with volume Δs^*. We assume that $s_0^* \overline{\in} \sigma^*$.

We give each elementary region a corresponding ordinal number 1, 2,..., m. In any elementary space, say the jth, the function $f(s^*,s)p(s^*/s)$ can be assumed constant and equal to $f(s_j^*,s)p(s_j^*/s)$, where s_j^* is an arbitrary fixed value of the variable s^* in the jth elementary region. Then

$$\rho_e = \lim_{\substack{\Delta s^* \to 0 \\ m \to \infty}} \left\{ \sum_{j=1}^{m} f(s_j^*, s)\, p(s_j^*/s)\, \Delta s^* \right\}. \tag{2.33}$$

Designating X_j the region of space X corresponding to decision s_j^*, and keeping in mind that

$$p(s_j^*/s)\, \Delta s^* = \int_{X_j} p(x/s)\, dx, \tag{2.34}$$

we obtain

$$\rho_e = \lim_{\substack{\Delta s \to 0 \\ m \to \infty}} \left\{ \sum_{j=1}^{m} \int_{X_j} f(s_j^*, s)\, p(x/s)\, dx \right\}. \tag{2.35}$$

The latter is transformed directly into the expression in the right part of correspondence (2.32).

Based on (2.32) and (2.2),

$$\begin{aligned} \int_{\tilde{S}^*} f(s^*, s)\, \tilde{p}(s^*/s)\, ds^* &= f(s_e^*, s)\, P(s_e^*/s) + \int_{S^*} f(s^*, s)\, p(s^*/s)\, ds^* = \\ &= \int_{X_e} f(s_e^*, s)\, p(x/s)\, dx + \int_{X - X_e} f(s^*, s)\, p(x/s)\, dx = \\ &= \int_{X} f(s^*, s)\, p(x/s)\, dx. \end{aligned} \tag{2.36}$$

Accordingly, criterion (2.1) can be represented in the form

$$\rho = \int_X dx \int_{\tilde{S}} ds\, r(s^*, s)\, \tilde{p}(s)\, p(x/s) = \text{extremum}. \qquad (2.37)$$

We shall assume, as above, that the optimality criterion consists of maximizing the quantity ρ. Then as the oscillations x arrive the decision which must be made is $s^*_x (s^*_x \in \tilde{S}^*)$, for which

$$\int_{\tilde{S}} r(s^*_x, s)\, \tilde{p}(s)\, \Lambda(s)\, ds = \sup_{s^*} \left[\int_{\tilde{S}} r(s^*, s)\, \tilde{p}(s)\, \Lambda(s)\, ds \right], \qquad (2.38)$$

where $\sup[\phi(x,s^*)]$ designates the greatest value of the function $\phi(x,s^*)$ taken for a given x with respect to all possible s^*.

The solution obtained for the problem of defining an optimal system in terms of a criterion of the general form (2.1) can be written with the help of system operator (2.5) in a fashion analogous to that used for simplified lines:

$$s^* = s^*_x \text{ for } x \in X_x, \qquad (2.39)$$

where X_x -- the set x (defined for every value of s^*_x from decision space $\tilde{S}^*$ on which equation (2.38) is satisfied.

The attempt to use the general solution of problem (2.38) encounters the same difficulties which were mentioned in the preceding section. Accordingly, it is again necessary to employ special types of criteria based on certain assumptions. A large number of more or less sensible optimality criteria can be proposed for the detection process in a real line. We shall limit ourselves to two criteria, which are continual analogs of the Neumann-Pearson and ideal observer criteria. The physical meaning of the criteria is retained, and by substituting (2.3) they are transformed to the corresponding optimality criteria for a simplified line.

The Neumann-Pearson criterion, which equipment developers consider to be most applicable, can be defined in generalized form by the equation

$$\rho = \Lambda_0 P(s_0^*/s_0) + \iint_{SS^*} \delta(s - s^*)\, p(s^*/s)\, ds\, ds^* = \max \tag{2.40}$$

assuming that

$$\int_{X_0} p(x/s_0)\, dx = 1 - F. \tag{2.41}$$

The nature of the optimality of criterion (2.40) is that for a given false alarm probability F the mean conditional probability density of making a correct decision is maximized: s*=s

$$\frac{1}{\int_S ds} \int_S |p(s^*/s)|_{s^*=s}\, ds = \max.$$

Based on the above proof, and keeping in mind that s*=s, (2.40) can be transformed into

$$\rho = \int_{X_0} \Lambda_0 p(x/s_0)\, dx + \int_{X-X_0} p(x/s)\, dx = \max \tag{2.42}$$

and the decision making rule formulated as follows. The decision s_0^* is made if for all $s \in S$

$$\Lambda(s) < \Lambda_0. \tag{2.43}$$

The critical level of the likelihood ratio Λ_0 is selected such that the probability that inequality (2.43) is satisfied assuming realization of the event s_0 is exactly equal to the assigned value 1-F.

The decision s_x^* that a signal with parameters s_x is present is made if

$$\Lambda(s_x) \geqslant \Lambda_0,$$
$$\Lambda(s_x) = \sup_s [\Lambda(s)]. \tag{2.44}$$

Analogously, we define the ideal observer criterion for a real line

$$\rho = P(s_0) P(s_0^*/s_0) + \iint_{SS^*} \delta(s^* - s) p(s) p(s^*/s) ds ds^* =$$
$$= \int_{X_0} P(s_0) p(x/s_0) dx + \int_{X-X_0} p(s) p(x/s) dx = \max. \tag{2.45}$$

The nature of the optimality of this criterion is that the mean (with respect to the a priori distribution) probability of making a correct decision is maximized. With a uniform a priori probability distribution criterion (2.45) again leads to expressions of the type (2.43) and (2.44).

During the radar observation process the situation is often encountered in which the signal (target) is detected reliably, and it is required only to provide the most accurate possible estimate of the parameters of the detected signal. Obviously, in this case it is best to use the estimate for the minimum mean square error, which is designated δ below, setting

$$r(s^*, s) = (s^* - s)^2 \quad и \quad P(s_0) = 0. \tag{2.46}$$

in (2.1).

In this case optimality criterion (2.1) takes on the form

$$\rho = \delta^2 = \int_S p(s) ds \int_{S^*} (s^* - s)^2 p(s^*/s) ds^* = \min \tag{2.47}$$

or

$$\rho = \delta^2 = \int_X p(x) dx \int_S (s - s^*)^2 p(s/x) ds = \min, \tag{2.48}$$

where p(s/x) is the conditional probability density for the set of useful parameters s, which is called the a posteriori probability function

$$p(s/x)=\frac{p(s)\,p(x/s)}{p(x)}=\frac{p(s)\,p(x/s_0)\,\Lambda(s)}{p(x)}. \qquad (2.49)$$

According to (2.48), when oscillation x arrives at the system input a decision s_x^* must be made which satisfies the equation

$$\int_S (s-s_x^*)^2\, p(s/x)\, ds=\inf_{s^*}\left[\int_S (s-s^*)^2\, p(s/x)\, ds\right]. \qquad (2.50)$$

The number $\inf_{s^*}[\phi(x,s^*)]$ indicates the smallest value of the function $\phi(x,s^*)$ taken for a given x with respect to all possible s*.

2.5. Discussion of Results and Definition of Object of Investigation

In all cases, regardless of whether a real or simplified line is employed, and regardless of the optimality criterion selected, the optimum receiver output can consist of the likelihood coefficient $\Lambda(s)$ or any other function of the set of useful parameters Y(s) which are mutually unique with respect to $\Lambda(s)$. Furthermore, the circuit of the optimum receiver is always defined by identifying the operations required to form from the received oscillation x(t) an arbitrary mutually unique likelihood coefficient function. In particular, we can emphasize that the problems of designing a receiver for maximum accuracy [criterion (2.47)] and for maximum detection reliability [criteria (2.21), (2.27), (2.40) and (2.45)] are identical. We also note that for simplified lines the output $\Lambda(s)$ is sufficient, while the necessary optimum output of the receiver is a discrete sequence of values of $\Lambda(s)$ for fixed values of the parameter s: $s_1, s_2, \ldots, s_m$.

The decision making rule in the decision device is defined for the selected receiver output, and depends upon a number of factors: the type of line (simplified or real), the a priori probability distribution $\tilde{p}(s)$ and the optimality criterion selected. It can be stated that by freeing the receiver to a substantial extent of the need for making allowance for initial assumptions, we transfer all of the responsibility

for their selection and consideration to the decision making rule in the decision device.

The decision making rule for a real line consists of the following for the cases of greatest practical interest [criteria (2.40) and (2.45)]. The critical level of the likelihood coefficient Λ_0 (or critical level of the output Y_0) is established. If the likelihood coefficient $\Lambda(s)$ in the working interval of S does not exceed Λ_0 at any point, it is decided that no signal is present. If $\Lambda(s)$ exceeds Λ_0 on some set of points, it is decided that a signal is present with parameters s_x, corresponding to the point with the highest value of the function $\Lambda(s)$, i.e., the maximum likelihood point. This decision making scheme during the detection process can be called a threshold-type scheme, since some function of the set of non-null decisions is compared with a threshold (critical number).

When the general optimality criterion (2.1) is employed, the decision making rule can also be reduced to a threshold scheme. In order to do this, we must form the function $\Sigma(s^*)$ of the set of non-null decisions

$$\Sigma(s^*) = \int_S [r(s^*, s) - r(s_0^*, s)]\, p(s)\, \Lambda(s)\, ds + P(s_0)\, r(s^*, s_0) \tag{2.51}$$

and define the critical number Σ_0

$$\Sigma_0 = P(s_0)\, r(s_0^*, s_0). \tag{2.52}$$

A decision is made that no signal is present when $\Sigma(s^*) \leq \Sigma_0$ on the set of all non-null decisions. Otherwise it is decided that a signal is present with parameters corresponding to the point S_x^* at which the function $\Sigma(s^*)$ reaches its greatest value.

If the cost of a miss for all signals is the same

$$r(s_0^*, s) = \begin{cases} r_{0s}, & s \neq s_0 \\ r_{00}, & s = s_0, \end{cases}$$

and the risk of all false alarms is the same

$$r(s^*, s_0) = \begin{cases} r_{s0}, & s^* \neq s_0^*, \\ r_{00}, & s^* = s_0^*, \end{cases}$$

it is best to employ, instead of (2.51) and (2.52),

$$\Sigma(s^*) = \int_S [r(s^*, s) - r_{0s}]\, p(s)\, \Lambda(s)\, ds$$

and

$$\Sigma_0 = [r_{00} - r_{s0}]\, P(s_0).$$

Simplified lines are a special case [with condition (2.3)] of real lines. Therefore, the decision algorithm in simplified lines can be obtained from the corresponding decisions for real lines by using (2.3).

Changes in specific operating conditions lead to changes in the a priori probabilities $P(s_0)$ and $p(s)$, and may also be accompanied by change in the risk function $r(s^*,s)$. Therefore, when the operating conditions change the threshold level Σ_0 must be changed, and perhaps certain other parameters of the decision devices employed in automatic radars as well. In non-automatic radars the information output by the receiver is passed directly to a human observer, rather than to a decision computing device. The operator makes independent allowance for specific operating conditions (risk function and a priori probabilities) in accordance with his experience and training. The study [16] cites a typical example of how a ship's radar operator will approach the evaluation of the information he receives differently depending upon whether the radar is operating on a stormy night near the shore or on a clear day in the open ocean. It is important for the radar designer that the definition of the optimum receiver remain the same in this case as well. In other words, the circuit of an optimum receiver does not depend only upon specific operating conditions, but also upon whether the radar is automatic and whether decisions are made by a human observer.

It is assumed in general below that the system incorporates a decision device which operates on the basis of the generalized Neumann-Pearson criterion, although this criterion is not the best one for all operating conditions.

When the parameters of the detected signal are estimated on the basis of the minimum mean square error criterion, the decision rule is to use as

the estimate, as follows from (2.50), the "center of gravity" point of the a posteriori probability function $p(s/x)$ or the function $p(s)\Lambda(s)$. For this reason, the signal parameter estimates obtained for detection based on the Neumann-Pearson and ideal observer criteria have the minimum possible mean square error only when the maximum likelihood point coincides with the "center of gravity" of the function $p(s)\Lambda(s)$.

We have represented the radar observation process as an experiment whose outcome is random, which makes it necessary to choose between competing hypotheses and to estimate certain parameters of the observed set of random quantities x. The subject of our study will be optimal circuits and those performance indicators of radar systems whose theoretically limiting values are bounded by the statistical nature of the problem resulting from the occurrence of interference. These performance indicators include the threshold signal or threshold signal/noise ratio, the accuracy with which the useful signal parameters are estimated and the resolution.

The threshold signal/noise ratio is the minimum signal/noise ratio required for the system to operate with the required reliability. The quantitative measure of reliability is, in general, the mathematical expectation ρ of the risk function r. When the Neumann-Pearson criterion is employed, this measure is the maximized mean correct detection probability and a fixed false alarm probability.

The accuracy with which useful signal parameters are estimated is measured in terms of the mean square error or dispersion of the estimates.

Resolution is estimated by the dimensions of the region in the useful parameter space S such that it is practically impossible to resolve two signals with useful parameters belonging to that region.

Statistical interpretation of the radar observation process makes it possible in principle to discuss and solve a broad group of problems involved in radar theory. These problems include the following:

1. Optimum methods for processing received oscillations in receiver (optimum receiver circuit).

2. Optimum decision making rule (optimum decision device circuit).

3. Quantitative estimate of potential capabilities of radar systems in terms of basic performance indicators.

4. Influence of parasitic parameters on potential capabilities of system and on optimum processing circuit.

5. Rational choice of radar signal form.

6. Influence on system performance indicators of deviations from optimum of processing method and decision making rule.

The investigation below is limited to examining the most practically important case -- that of reception against the background of additive random interference. The analytical expressions for the functions p(x/s) and Λ(s) which figure in the calculations are defined as follows. The probability density p(x/s,u) of the quantity x, assuming that the signal з(t,s,u) is input, is obviously the same as the probability density of the interference p(n) if the value of the interference n(t) is assumed to be

$$n(t)=x(t)-з(t,s,u). \tag{2.53}$$

Thus, for the class of random interference whose probability distribution is defined through (1.79),

$$p(x/s,u)=k\exp\left\{-\frac{1}{N_0}\int_{t_1}^{t_2}[x(t)-з(t,s,u)]^2\,dt\right\}, \tag{2.54}$$

where (t_1,t_2) is an arbitrary interval which includes the time interval on which the function з(t;s;u) is non-zero.

Analogously, assuming the realization of the event s_0,

$$p(x/s_0)=k\exp\left\{\frac{1}{N_0}\int_{t_1}^{t_2}x^2(t)\,dt\right\}, \tag{2.55}$$

and the likelihood coefficient Λ(s,u) for the set of useful s and parasitic u parameters of the signal will be

$$\Lambda(s, u) = \frac{p(x/s, u)}{p(x/s_0)} = \exp\left\{-\frac{1}{N_0}\int_{t_1}^{t_2} s^2(t, s, u)\,dt + \right.$$
$$\left. + \frac{2}{N_0}\int_{t_1}^{t_2} x(t)\,s(t, s, u)\,dt\right\}. \tag{2.56}$$

The likelihood function p(x/s) and the likelihood coefficient $\Lambda(s)$ of the set of useful parameters are obtained by statistical averaging of (2.54) and (2.56) with respect to all possible values of u

$$p(x/s) = \int p(u)\,p(x/s, u)\,du, \tag{2.57}$$

$$\Lambda(s) = \frac{p(x/s)}{p(x/s_0)} = \int p(u)\,\Lambda(s, u)\,du. \tag{2.58}$$

We note that according to (2.58) it is completely impossible to relieve the receiver of the need for allowing for a priori data, since formation of the likelihood coefficient $\Lambda(s)$ requires that the a priori distribution of the parasitic parameters p(u) be available or given.

In order to simplify the investigation and establish separately the influence of each of the factors on the radar observation process, we shall begin our study with the simplest cases, and gradually make the circuit more complex as we draw nearer to real operating conditions.

Chapter 4. Estimation of Signal Parameters

4.1. Discussion of Problem and Methods for Solution

The problem of estimating signal parameters, which arises in radar as well as related areas of electrical engineering -- radio navigation, radio remote control, radio telemetry, among others -- is formulated as follows. It is well known that the oscillation x(t) which arrives at the input of a receiver consists of an additive mixture of interference n(t) and signal s(t,s,u), the set of useful parameters of which s belongs to the interval S. Given, or defined by introducing certain assumptions, are the a priori distribution functions of the useful and parasitic parameters p(s) and p(u), as well as the probability density functional p(n) of the interference n(t). The problem calls for defining the system operator (processing circuit and decision making rule) which guarantees the best (in terms of mean square) estimate, and calculating the numerical characteristics of the parameter sampling accuracy -- the mean square errors.

One more initial condition must be added to this general statement of the problem for the case of sampling the parameters of radar signals. The problem of the most accurate possible sample of the signal parameter arises after the fact that a target is present has been registered reliably. We can therefore assume that when the parameters are sampled the signal is significantly stronger than the interference ($q^2 \gg 1$), at least by enough for reliable target detection. We note that there is no limit placed on the input signal/noise power ratio.

The latter condition is significant. The output Y(s) on the set of possible decisions contains a large number of isolated maxima (spikes), one of which is caused by the signal-noise mixture, with the remaining being caused by interference alone. If the signal energy is too low, there is ambiguity in the detection and sampling -- any large interference spikes on the interval S may be interpreted as a spike caused by a signal, since the presence of a signal has been stipulated in advance. Under these conditions, it is meaningless to solve the problem of radar signal parameter estimation accuracy.

The problem of estimating signal parameters and the problem of signal detection have a common statistical model, which was described in §2.1. The difference between the estimation problem and the detection problem, which also accompanies parameter estimation, consists of differences in the risk function and in the initial assumptions with respect to the a priori probabilities, which essentially reflects a difference in the tactical content of both problems. The problem of parameter estimation is usually associated with the target tracking process or with the process of observing the trajectory of a detected target.

The second chapter examined the problem of parameter estimation in very general features. The optimal system operator was defined, and analytical formulas (2.47) and (2.48) for mean square errors were presented. If the estimate of s* is obtained in expressions (2.47) and (2.48) by applying the optimum operator, these expressions determine the theoretically limiting or potential mean square errors.

The objective of the continued investigation in the present chapter is a) to establish certain general regularities which occur in estimating radar signal parameters; b) to analyze potential mean square errors; c) to define a method for constructing practical circuits for obtaining best estimates.

For definition we shall assume that the set of useful parameters s consists of the two scalar parameters α and β

$$s=(\alpha,\ \beta) \qquad \text{and} \qquad ds=d\alpha d\beta. \tag{4.1}$$

Instead of the mathematical expectation of the square of the absolute value of the error vector defined through (2.47) and (2.48) we shall, in accordance with practical requirements, be examining the mean square errors $\langle\delta_\alpha^2\rangle$ and $\langle\delta_\beta^2\rangle$ of the estimate of each of the scalar parameters α and β. The optimum system must simultaneously minimize the mean square values of both of the error vector components $\alpha-\alpha^*$ and $\beta-\beta^*$ (α^* and β^* -- estimates of the parameters α and β), i.e., it must simultaneously minimize $\langle\delta_\alpha^2\rangle$ and $\langle\delta_\beta^2\rangle$. When necessary, the discussions below can be extended to sets of large numbers of useful parameters.

There are at least two possible different methods for calculating mean square errors. The first method is based on representation (2.47), in accordance with which the mean square errors $\langle\delta_\alpha^2\rangle$ and $\langle\delta_\beta^2\rangle$ are defined by the formulas

$$\begin{aligned}\langle\delta_\alpha^2\rangle&=\int_S p(s)\,ds\int_{S^*} p(s^*/s)\ (\alpha^*-\alpha)^2\,ds^*,\\ \langle\delta_\beta^2\rangle&=\int_S p(s)\,ds\int_{S^*} p(s^*/s)\ (\beta^*-\beta)^2\,ds^*.\end{aligned} \tag{4.2}$$

The structure of the set S^* is the same as that of set S. The analysis is done as follows. An ensemble of cases is examined in which the received oscillation $x(t)$ represents the sum of interference $n(t)$ and signal with fixed values of the useful parameters α and β or s. The estimate $s^*=(\alpha^*,\beta^*)$ obtained by applying some operator, not necessarily optimum, is random. The conditional mean square errors are first defined for a fixed value of $s=(\alpha,\beta)$

$$\begin{aligned}[\delta_\alpha^2]_s&=\int_{S^*}(\alpha^*-\alpha)^2\,p(s^*/s)\,ds^*,\\ [\delta_\beta^2]_s&=\int_{S^*}(\beta^*-\beta)^2\,p(s^*/s)\,ds^*,\end{aligned} \tag{4.3}$$

after which averaging is done with respect to s

$$\langle\delta_\alpha^2\rangle = \int_S p(s)[\delta_\alpha^2]_s ds, \quad \langle\delta_\beta^2\rangle = \int_S p(s)[\delta_\beta^2]_s ds. \tag{4.4}$$

If for all s ∈ S the errors $[\delta_\alpha^2]_s$ and $[\delta_\beta^2]_s$ are independent of s, then

$$\langle\delta_\alpha^2\rangle = [\delta_\alpha^2]_s, \quad \langle\delta_\beta^2\rangle = [\delta_\beta^2]_s \tag{4.5}$$

and there is no need to average in (4.4).

The result, of course, remains unchanged if, fixing both the useful parameters s and the parasitic parameters u, we find

$$[\delta_\alpha^2]_{su} = \int_{S^*} (\alpha^* - \alpha)^2 p(s^*/s, u) ds^*,$$

$$[\delta_\beta^2]_{su} = \int_{S^*} (\beta^* - \beta)^2 p(s^*/s, u) ds^*,$$

after which we average with respect to s and u.

The second method for analyzing mean square errors is based on representation (2.48), according to which

$$\langle\delta_\alpha^2\rangle = \int_X p(x) dx \int_S (\alpha - \alpha^*)^2 p(s/x) ds,$$

$$\langle\delta_\beta^2\rangle = \int_X p(x) dx \int_S (\beta - \beta^*)^2 p(s/x) ds. \tag{4.6}$$

An ensemble of cases is examined in which, regardless of the useful parameters s = (α,β) of the "transmitted" signal, the input of the receiving device receives the same oscillation s(t), so that the same estimate s*=(α*, β*) is obtained. The conditional mean square errors are first determined for fixed values of x and s*

$$[\delta_\alpha^2]_x = \int_S (\alpha - \alpha^*)^2 \, p(s/x)\, ds,$$
$$[\delta_\beta^2]_x = \int_S (\beta - \beta^*)^2 \, p(s/x)\, ds, \tag{4.7}$$

after which we average with respect to x

$$\langle\delta_\alpha^2\rangle = \int_X p(x)[\delta_\alpha^2]_x\, dx,\ \langle\delta_\beta^2\rangle = \int_X p(x)[\delta_\beta^2]_x\, dx. \tag{4.8}$$

If the errors $[\delta_\alpha^2]_x$ and $[\delta_\beta^2]_x$ are independent of x,

$$\langle\delta_\alpha^2\rangle = [\delta_\alpha^2]_x;\ \langle\delta_\beta^2\rangle = [\delta_\beta^2]_x \tag{4.9}$$

and it becomes unnecessary to average in (4.8). It was essentially this statement for which Kotel'nikov [18] solved the problem for signals containing one useful parameter and no parasitic parameters. This analytical method is also employed in [8, 15], but without sufficient foundation. The investigation below shall employ both methods for calculating mean square errors. In practical calculations preference will be given to the first method, which produces comparatively simple analytical expressions.

In some studies [31] the calculation of limiting accuracy reduces to calculating the dispersion of the effective estimates. Kramer [19] presents a proof that when there is one scalar random parameter α the mean square error in its estimate, corresponding to the quantity $[\delta_\alpha^2]_s$, which we defined by means of (4.3), is bounded from below by the number

$$\delta^2 \geqslant \frac{1}{\int_X \left[\frac{\partial}{\partial a} \ln p(x/a)\right]^2 p(x/a)\, dx}. \tag{4.10}$$

It is assumed that the derivative figuring in formula (4.10) exists, and that the estimate is unbiased, i.e.,

$$\langle a^{*}\rangle = a, \tag{4.11}$$

and that the error dispersion can be examined instead of the mean square errors.

The estimate α^* for which the dispersion reaches the smallest possible value defined by the formula cited above is called the effective estimate. An effective estimate exists [i.e. (4.10) as an equals sign] when and only when these two conditions are satisfied:

$$\text{a) } p(x/a) = \varphi(x)\, p(a^{*}/a), \tag{4.12}$$

where p(a*/a) is the distribution function of the estimates for a given fixed value of α, $\phi(x)$ is an arbitrary function of x;

$$\text{б) } \frac{\partial}{\partial a} \ln (p(a^{*}/a) = k(a^{*} - a), \tag{4.13}$$

where k is a coefficient independent of α^*, but which may depend upon α.

When there is a large number of random parameters α, β,... there are analogous, but more complicated, formulas for the dispersion of the effective estimate and for the existence condition of effective estimates. It follows from the very nature of conditions a) and b) that effective estimates can be expected to exist only in certain special cases. Therefore, when investigating an arbitrary system intended for sampling random parameters, analysis of the dispersion of the effective estimates by formulas analogous to (4.10) still do not provide the values of the dispersion of the actual estimates, or even the theoretically limiting dispersion of the estimates for the system in question.

The estimates (α^*, β^*) can be obtained through various methods which, in the general case, yield different deviations from the theoretically limiting estimates. The most important of the possible estimation methods

is the maximum likelihood method, according to which the estimate (α_0^*, β_0^*) is the point at which the a posteriori probability function $p(\alpha, \beta/x)$ or, if the a posteriori distribution of $p(\alpha,\beta)$ can be assumed uniform, the likelihood coefficient $\Lambda(\alpha,\beta)$, reaches its maximum value.

The estimation method formulated above coincides with the decision making rule presented in §2.4. This method is important because in all cases in which an effective estimate exists it can be obtained by the maximum likelihood method [19]. Another advantage of the maximum likelihood method over many others is the fact that the maximum likelihood point is invariant with respect to arbitrary mutually unique transformation of the output. This makes it easier to construct the processing circuit for obtaining the maximum likelihood estimate.

4.2. General Principles of Parameter Estimation

In order to study some of the principles occurring in the estimation of radar signal parameters, we shall employ a method for calculating mean square errors in which an ensemble of cases with a fixed value of the received oscillation $x(t)$ is examined.

The a posteriori probability density for the useful parameters $p(\alpha,\beta/x)$ is

$$p(\alpha, \beta/x) = k_x\, p(\alpha, \beta)\, \Lambda(\alpha, \beta), \tag{4.14}$$

where $p(\alpha,\beta)$ and $\Lambda(\alpha,\beta)$ represent the a priori probability density and likelihood coefficient for the useful parameters, and k_x is the proportionality coefficient, which in general depends upon the received oscillation $x(t)$. The numerical value of the coefficient k_x can be found from the normalization conditions

$$\iint_S p(\alpha, \beta/x)\, d\alpha\, d\beta = 1. \tag{4.15}$$

When estimating signal parameters under reliable detection conditions we are interested in the behavior of the function $p(\alpha,\beta/x)$, and consequently the function $p(\alpha,\beta)$, only in the vicinity of a powerful spike caused by the

presence of a signal. We shall call this region the vicinity of the estimate and designate it σ. Obviously, it can almost always be assumed that the distribution of $p(\alpha,\beta)$ is near-uniform in the vicinity of the estimate. Accordingly, we shall assume below

$$p(\alpha, \beta/x) = k_x' \Lambda(\alpha, \beta). \tag{4.16}$$

When a priori information is available which can modify a posteriori distribution (4.16) significantly, it must be allowed for by substituting the basic expression (4.14) for (4.16).

A decision device based on studying the output $Y(\alpha,\beta)$, which is a mutually unique function of the a posteriori distribution (4.16), outputs in some fashion the estimate α^*, β^* obtained. The conditional mean square errors for given values of $x(t)$ and (α^*, β^*), according to (4.7), are

$$\begin{aligned} [\delta_\alpha^2]_x &= \iint_S (\alpha - \alpha^*)^2 \cdot p(\alpha, \beta/x)\, d\alpha d\beta, \\ [\delta_\beta^2]_x &= \iint_\sigma (\beta - \beta^*)^2\, p(\alpha, \beta/x)\, d\alpha d\beta. \end{aligned} \tag{4.17}$$

In examining expressions (4.17), we can reach some practical conclusions. If the signal contains several random parameters, such as the parameter α and others, the mean square error in the estimate of the parameter α is independent of whether this parameter is estimated in conjunction with the others, or if all of the other parameters are perceived by the system as parasitic and information regarding them is destroyed during the processing (by integration).

For every given value of x the minimum possible value of the errors $[\delta_\alpha^2]_x$ and $[\delta_\beta^2]_x$, and consequently of the mean square errors, (4.8), is obtained when the estimates of the parameters are the coordinates of the "center of gravity" α_{cg}, β_{cg} of the a posteriori distribution $p(\alpha,\beta/x)$. The latter,

of course, is the same as the result obtained in the second chapter, if we use as our basis the minimum mean square of the absolute value of the error vector.

The best estimate of the useful parameters, which has potential mean square error, is thus the estimate with respect to the "center of gravity" of the a posteriori distribution.

The estimate α_0^*, β_0^* obtained by the maximum likelihood method, in the general case, in contrast to the estimate α_{cg}, β_{cg}, is biased, and has mean square errors which exceed the potential values. The maximum likelihood estimate is an unbiased best estimate only when the coordinates of the center of gravity and the maximum likelihood points of the a posteriori distribution of $p(\alpha,\beta/x)$ coincide

$$\alpha_0^* = \alpha_{cg}, \ \beta_0^* = \beta_{cg}. \tag{4.18}$$

The sufficient condition for ensuring equality (4.18) is the condition of symmetry on the plane (α,β) of the lines of the level of function $p(\alpha,\beta/x)$ with respect to the maximum likelihood point (α_0^*, β_0^*). This condition can be written analytically as

$$p(\alpha_0^* + \hat{\alpha}, \beta_0^* + \hat{\beta}/x) = p(\alpha_0^* - \hat{\alpha}, \beta_0^* - \hat{\beta}/x). \tag{4.19}$$

In fact, when (4.19) is satisfied

$$\alpha_{cg} = \iint_S \alpha p(\alpha, \beta/x)\, d\alpha d\beta =$$
$$= \alpha_0^* + \int_{-a}^{a} \hat{\alpha}\, d\hat{\alpha} \int_{-b}^{b} p(\alpha_0^* + \hat{\alpha}, \beta_0^* + \hat{\beta}/x)\, d\hat{\beta} = \alpha_0^*,$$

since the function

$$\int_{-b}^{b} p(\alpha_0^* + \hat{\alpha}, \beta_0^* + \hat{\beta}/x)\, d\hat{\beta} = \int_{-b}^{b} p(\alpha_0^* - \hat{\alpha}, \beta_0^* - \hat{\beta}/x)\, d\hat{\beta} =$$
$$= \int_{-b}^{b} p(\alpha_0^* - \hat{\alpha}, \beta_0^* + \hat{\beta}/x)\, d\hat{\beta}$$

is an even function of α. The symbols a and b define the intervals of variation of the increments $\hat{\alpha}$ and $\hat{\beta}$ of the parameters α and β, respectively. The equality of β_0^* and β_{cg} is obtained analogously under condition (4.19)

If the a posteriori probability function $p(\alpha,\beta/x)$ in the vicinity of the maximum likelihood estimate (α_0^*, β_0^*) is analytical, condition (4.19) means that it can be represented as the series

$$p(\alpha, \beta/x) = \sum_{i,j} a_{i,j} (\alpha - \alpha_0^*)^i (\beta - \beta_0^*)^j; \quad (i + j = 2n;\ i,\ j,\ n = 0,\ 1,\ 2, \ldots). \tag{4.20}$$

All of the coefficients $a_{i,j}$ of series (4.20) for which the sum of the powers i+j is an odd number are zero. Representation (4.20) follows directly from the fact that according to (4.19)

$$p(\alpha, \beta/x) = \frac{1}{2} \{ p[\alpha_0^* + (\alpha - \alpha_0^*),\ \beta_0^* + (\beta - \beta_0^*)/x] + p[\alpha_0^* - (\alpha - \alpha_0^*),\ \beta_0^* - (\beta - \beta_0^*)/x] \}. \tag{4.21}$$

We note that in conditions (4.19) and (4.20) above, which are sufficient for the maximum likelihood point to coincide with the center of gravity of the distribution of $p(\alpha,\beta/x)$, the a posteriori probability function can be replaced with any other function which is unique with respect to $p(\alpha,\beta/x)$. These conditions, when necessary, can be extended to the case of a large number of useful parameters. Further examination of the question of the potential accuracy of sampling of signal parameters requires more detailed study of the a posteriori probability function.

Of greatest interest for the practical applications encountered below is the case in which the signal contains, besides the useful parameters α and β, two parasitic random parameters: the initial phase ϕ and intensity ε, i.e., when the received signals are of the form

$$\varepsilon s(t, \alpha, \beta, \varphi) = \varepsilon \operatorname{Re}\{S(t, \alpha, \beta) e^{j(2\pi f_0 t + \varphi)}\}. \tag{4.22}$$

We also assume that the interference has the nature of white noise, or of a normal process with a very wide spectrum, so that the probability density functional of the interference p(n) can be represented by means of (1.79).

The a posteriori probability function for all random parameters is

$$p(\alpha, \beta, \varepsilon, \varphi/x) = k_x \frac{p(\varepsilon)}{2\pi} \exp\left[-\frac{\varepsilon^2}{N_0}\int_{-\infty}^{\infty} s^2(t, \alpha, \beta, \varphi)\,dt + \right.$$
$$\left. + \frac{2\varepsilon}{N_0}\int_{-\infty}^{\infty} x(t) s(t, \alpha, \beta, \varphi)\,dt\right], \tag{4.23}$$

or

$$p(\alpha, \beta, \varepsilon, \varphi/x) = k_x \frac{p(\varepsilon)}{2\pi} \exp\left\{-\varepsilon^2 q^2(\alpha, \beta) + \right.$$
$$+ \frac{2\varepsilon}{N_0}\int_{-\infty}^{\infty} [x(t) - \varepsilon_0^* s(t, \alpha_0^*, \beta_0^*, \varphi_0^*)] s(t, \alpha, \beta, \varphi)\,dt +$$
$$\left. + \frac{2\varepsilon\varepsilon_0^*}{N_0}\int_{-\infty}^{\infty} s(t, \alpha_0^*, \beta_0^*, \varphi_0^*) s(t, \alpha, \beta, \varphi)\,dt\right\}, \tag{4.24}$$

where $(\alpha_0^*, \beta_0^*, \phi_0^*, \varepsilon_0^*)$ is the maximum likelihood point, found from the system of equations:

$$
\begin{aligned}
&\ln p(\alpha,\ \beta,\ \varepsilon,\ \varphi/x)=\int_{-\infty}^{\infty}[x(t)-\varepsilon_0^*\,\text{з}(t,\ \alpha_0^*,\ \beta_0^*,\ \varphi_0^*)]\times\\
&\qquad\times\frac{\partial}{\partial\alpha}\text{з}(t,\ \alpha_0^*,\ \beta_0^*,\ \varphi_0^*)\,dt=0;\\
&\frac{\partial}{\partial\beta}\ln p(\alpha,\ \beta,\ \varepsilon,\ \varphi/x)=\int_{-\infty}^{\infty}[x(t)-\varepsilon_0^*\,\text{з}(t,\ \alpha_0^*,\ \beta_0^*,\ \varphi_0^*)]\times\\
&\qquad\times\frac{\partial}{\partial\beta}\text{з}(t,\ \alpha_0^*,\ \beta_0^*,\ \varphi_0^*)\,dt=0,\\
&\frac{\partial}{\partial\varphi}\ln p(\alpha,\ \beta,\ \varepsilon,\ \varphi/x)=\int_{-\infty}^{\infty}[x(t)-\varepsilon_0^*\,\text{з}(t,\ \alpha_0^*,\ \beta_0^*,\ \varphi_0^*)]\times\\
&\qquad\times\frac{\partial}{\partial\varphi}\text{з}(t,\ \alpha_0^*,\ \beta_0^*,\ \varphi_0^*)\,dt=0,\\
&\frac{\partial}{\partial\varepsilon}\ln p(\alpha,\ \beta,\ \varepsilon,\ \varphi/x)=\int_{-\infty}^{\infty}[x(t)-\varepsilon_0^*\,\text{з}(t,\ \alpha_0^*,\ \beta_0^*,\ \varphi_0^*)]\times\\
&\qquad\times\text{з}(t,\ \alpha_0^*,\ \beta_0^*,\ \varphi_0^*)\,dt=0.
\end{aligned}
\tag{4.25}
$$

The second term in the braces in (4.24) can be omitted in the vicinity of $(\alpha_0^*,\ \beta_0^*)$. This possibility is determined by two factors. First of all, the function $[x(t) - \varepsilon_0^*\ \text{з}(t,\ \alpha_0^*,\ \beta_0^*,\ \phi_0^*)]$ represents interference with the minimum possible energy for given x(t) superimposed on the signal. Accordingly, the mean square value of the second term for any α and β does not exceed $\sqrt{2}\,q$, and under reliable detection conditions is significantly smaller than the amplitude value of the third term near (α_0^*,β_0^*).

Second, the second term in (4.24) and all of its first-order partial derivatives at the point $(\alpha_0^*, \beta_0^*, \phi_0^*)$ are zero according to (4.25). Furthermore, all of the higher-order partial derivatives with respect to ϕ are zero as well, since

$$\frac{\partial^{2n+k}}{\partial\varphi^{2n+k}} s(t, \alpha, \beta, \varphi) = (-1)^n \frac{\partial^k}{\partial\varphi^k} s(t, \alpha, \beta, \varphi); \quad (n, k = 0, 1, 2, \ldots). \tag{4.26}$$

Therefore, when the second term in (4.24) is expanded into a series arranged by powers of $(\alpha-\alpha_0^*)$, $(\beta-\beta_0^*)$, $(\phi-\phi_0^*)$ only those terms are retained which contain the higher powers of the small parameters $(\alpha-\alpha_0^*)$ and $(\beta-\beta_0^*)$. Thus, the second term in (4.24) at the point (α_0^*, β_0^*) is equal to zero, and is very small in the vicinity of that point (the mean square value is significantly smaller than $\sqrt{2}\,q$). Furthermore, it is approximately the case that

$$p(\alpha, \beta, \varepsilon, \varphi/x) = k_x \frac{p(\varepsilon)}{2\pi} \exp\left[-\varepsilon^2 q^2(\alpha, \beta) + \frac{2\varepsilon\varepsilon_0^*}{N_0} \int_{-\infty}^{\infty} s(t, \alpha, \beta, \varphi)\, s(t, \alpha_0^*, \beta_0^*, \varphi_0^*)\, dt\right]. \tag{4.27}$$

Elimination of the parasitic parameters yields

$$p(\alpha, \beta, \varepsilon/x) = k_x p(\varepsilon) \exp[-\varepsilon^2 q^2(\alpha, \beta)] \times I_0\left[\frac{\varepsilon\varepsilon_0^*}{N_0} \left| \int_{-\infty}^{+\infty} \overline{S(t, \alpha, \beta)}\, S(t, \alpha_0^*, \beta_0^*)\, dt \right|\right], \tag{4.28}$$

$$p(\alpha, \beta/x) = k_x \Phi\left[q^2(\alpha, \beta), \varepsilon_0^* \left| \int_{-\infty}^{\infty} \overline{S(t, \alpha, \beta)}\, S(t, \alpha_0^*, \beta_0^*)\, dt \right|\right]. \tag{4.29}$$

where Φ -- a certain function determined by the intensity distribution $p(\varepsilon)$. For a Rayleigh distribution of the parameter ε

$$p(\alpha, \beta/x) = k_x \exp\left\{\frac{(\varepsilon_0^*)^2}{4[1+q^2(\alpha, \beta)]}\left[\left|\int_{-\infty}^{\infty} \overline{S(t, \alpha, \beta)} \times \right.\right.\right.$$
$$\left.\left.\left. \times S(t, \alpha_0^*, \beta_0^*)\, dt\right|\right]^2\right\}. \qquad (4.30)$$

Outside the region σ, which by definition is the vicinity of the estimate, expressions (4.29) and (4.30), of course, cannot serve as an approximation of the a posteriori distribution. However, under reliable detection conditions the probability concentrated outside the region σ under the functions (4.28)-(4.30), and under the corresponding exact expressions for the a posteriori probability function, is negligibly small. Therefore, in order to calculate the mean square errors by formulas (4.17) with good enough accuracy for practical applications, the coefficient k_x can be defined by substituting the expressions obtained for the function $p(\alpha,\beta/x)$ in the vicinity of the estimate in normalization condition (4.15). The coefficient k_x is uniquely defined as a function of the estimate $(\alpha_0^*, \beta_0^*, \varepsilon_0^*)$.

Our analysis establishes that the a posteriori probability function $p(\alpha,\beta/x)$, or any other optimal output $Y(\alpha,\beta)$ in the vicinity of the estimate σ always has the same appearance, i.e., it is always some known function ϕ

$$p(\alpha, \beta/x) = \varphi(\alpha, \beta, \alpha_0^*, \beta_0^*, \varepsilon_0^*) \qquad (4.31)$$

of the variables α, β and the estimates α_0^*, β_0^*, ε_0^*, and does not depend directly upon the received oscillation $x(t)$. For signals with fixed intensity the parameter ε_0^* in the latter expression, of course, must be omitted.

We conclude on the basis of (4.31) that in the general case the coordinates of the "center of gravity" α_{cg}, β_{cg} of the a posteriori distribution, which are the estimates with the theoretically limiting mean square deviation,

can be represented as certain known functions ϕ_1 and ϕ_2 of the maximum likelihood point $(\alpha_0^*, \beta_0^*, \varepsilon_0^*)$

$$\alpha_{cg} = \iint_S \alpha\varphi(\alpha, \beta, \alpha_0^*, \beta_0^*, \varepsilon_0^*)\, d\alpha d\beta = \varphi_1(\alpha_0^*, \beta_0^*, \varepsilon_0^*),$$
$$\beta_{cg} = \iint_S \beta\varphi(\alpha, \beta, \alpha_0^*, \beta_0^*, \varepsilon_0^*)\, d\alpha d\beta = \varphi_2(\alpha_0^*, \beta_0^*, \varepsilon_0^*). \tag{4.32}$$

Therefore, in the general case obtaining an estimate which realizes the potential capabilities of the system with respect to accuracy can be reduced to obtaining the maximum likelihood estimate of α_0^*, β_0^*, ε_0^*.
Furthermore, both the useful parameters and the intensity parameters are subject to estimation.

We note also that the dispersion of the estimates α_0^*, β_0^* is always the same as the dispersion of the best estimates α_{cg}, β_{cg}, since for every given value of x(t) it is determined by the dispersion of the a posteriori distribution $p(\alpha,\beta/x)$.

In the overwhelming majority of cases with which we must deal when sampling the parameters of radar signals, the absolute value of the integral $\int_{-\infty}^{\infty} \overline{S(t, \alpha, \beta)}\, S(t, \alpha + \hat{\alpha}, \beta + \hat{\beta})\, dt$ does not depend upon the parameters α and β, but is a function only of the increments $\hat{\alpha}$ and $\hat{\beta}$

$$\left| \int_{-\infty}^{\infty} \overline{S(t, \alpha, \beta)}\, S(t, \alpha + \hat{\alpha}, \beta + \hat{\beta})\, dt \right| = 2Q^2\Psi(\hat{\alpha}, \hat{\beta}). \tag{4.33}$$

Assuming in the left part of the latter equality that one time $\alpha=\beta=0$, and the second time $\alpha= -\hat{\alpha}$, $\beta= -\hat{\beta}$, and keeping in mind that under condition (4.33) both substitutions should produce the same result, we obtain

$$\Psi(\hat{\alpha}, \hat{\beta}) = \Psi(-\hat{\alpha}, -\hat{\beta}). \tag{4.34}$$

Equality (4.19) which, in turn, leads to (4.20), will hold for a posteriori probability functions (4.29) and (4.30). Thus, if the modulation function $S(t,\alpha,\beta)$ of the received signal for $(\alpha,\beta)\in\sigma$ obeys condition (4.33), the maximum likelihood estimate (α_0^*, β_0^*) is unbiased (coincides with the "center of gravity" α_{cg}, β_{cg}) and directly yields the theoretically limiting mean square deviation from the true values of the measured parameters.

For cases in which the signal exceeds the interference by a significantly greater amount ($q^2>>1$) the errors $(\alpha_0^*-\alpha_0)$ and $(\beta_0^*-\beta)$ in the sample of the signal parameters are small, and regardless of whether or not condition (4.53) is satisfied the optimum output $Y(\alpha,\beta)$ can be represented approximately as the series

$$Y(\alpha,\ \beta)=\ln p(\alpha,\ \beta/x)=a_{00}+a_{20}(\alpha-\alpha_0^*)^2+ \\ +a_{02}(\beta-\beta_0^*)^2+a_{11}(\alpha-\alpha_0^*)(\beta-\beta_0^*),\ (\alpha,\ \beta)\in\sigma, \tag{4.35}$$

which retains only the terms with low powers of the small parameters. The coefficients of the series are independent of α and β and, under condition (4.33), of α_0^* and β_0^* as well. The possibility of representing the function $\ln p(\alpha,\beta/x)$ in the form (4.35), which is equivalent to approximating the a posteriori probability function by a normal distribution, guarantees the existence of an effective unbiased estimate of the parameters of the received signal [19]. In other words, when the errors are essentially small the maximum likelihood estimate is nearly always an effective estimate.

Consequently, if condition (4.33) occurs or representation (4.35) can be used, the estimates obtained during detection based on the generalized Neumann-Pearson criterion or ideal observer criterion are the best estimates.

The conditional mean square errors $[\delta_\alpha^2]_x$ and $[\delta_\beta^2]_x$ calculated for a fixed value of $x(t)$, according to (4.31), depend upon α_0^*, β_0^*, ε_0^* in the general case. Consequently, calculation of the mean square errors $\langle\delta_\alpha^2\rangle$, $\langle\delta_\beta^2\rangle$ in the general case can be reduced to averaging $[\delta_\alpha^2]_x$ and $[\delta_\beta^2]_x$ with respect to α_0^*, β_0^*, ε_0^*. If condition (4.33) is satisfied, expression (4.31) for the a posteriori probability becomes the following:

$$p(\alpha,\ \beta/x)=\varphi(\varepsilon_0^*,\ \alpha-\alpha_0^*,\ \beta-\beta_0^*). \tag{4.36}$$

The conditional mean square errors $[\delta_\alpha^2]_x$ and $[\delta_\beta^2]_x$ of the best estimate depend only upon the parameter ε_0^*. Accordingly,

$$\langle\delta_\alpha^2\rangle=\int_0^\infty [\delta_\alpha^2]_x\, p(\varepsilon_0^*)\, d\varepsilon_0^*,\quad \langle\delta_\beta^2\rangle=\int_0^\infty [\delta_\beta^2]_x\, p(\varepsilon_0^*)\, d\varepsilon_0^* \tag{4.37}$$

and only for condition (4.33) and a fixed intensity of the received signals does equality (4.9) occur

$$\langle\delta_\alpha^2\rangle=[\delta_\alpha^2]_x,\quad \langle\delta_\beta^2\rangle=[\delta_\beta^2]_x.$$

Finally, an important conclusion which follows directly from representation (4.31) is that the obtaining of an estimate is not necessarily accompanied by reproduction of the output $Y(\alpha,\beta)$ for all values of α and β in region σ. Fundamentally, if the output is defined in the region σ at three points, such as (α_1, β_1), (α_2,β_2) and (α_3, β_3), the solution of the system of equations

$$\begin{aligned}
&\varphi(\alpha_1,\ \beta_1,\ \alpha_0^*,\ \beta_0^*,\ \varepsilon_0^*)=Y(\alpha_1,\ \beta_1),\\
&\varphi(\alpha_2,\ \beta_2,\ \alpha_0^*,\ \beta_0^*,\ \varepsilon_0^*)=Y(\alpha_2,\ \beta_2),\\
&\varphi(\alpha_3,\ \beta_3,\ \alpha_0^*,\ \beta_0^*,\ \varepsilon_0^*)=Y(\alpha_3,\ \beta_3)
\end{aligned} \tag{4.38}$$

makes it possible to find the estimate α_0^*, β_0^* and, when necessary, the parameter ε_0^*. When obtaining estimates by solving a system of equations here and below we assume, in accordance with real operating conditions, that the intensity of the received signal is a parasitic random parameter. Other quantities, such as the derivatives of the output, can be employed instead of the output in system (4.38). It is necessary only that for random intensity, which is perceived as a

parasitic parameter, the number of equations exceed by one the number of measured parameters and that the system have a unique solution in the vicinity of the estimate.

The investigation could have been done in more general form without specifying the physical nature of the parasitic parameters. However, the practical applications of the theory which are encountered below do not require this. In addition, this would have complicated the investigation significantly.

4.3. Analysis of Dispersion of Estimates Obtained by Maximum Likelihood Method

The preceding section indicated that the maximum likelihood estimate has theoretically limiting (potential) dispersion. There is no bias in the maximum likelihood estimate, since condition (4.33) is usually satisfied, or representation (4.35) can be used. However, even when the estimate is biased, allowance can be made for this during sampling. Therefore, the potential capabilities of the system in terms of accuracy can be determined by the scattering or dispersion of the estimates ($\alpha_0{}^*$, $\beta_0{}^*$) obtained by the maximum likelihood method.

Our next problem, accordingly, is to obtain more or less simple analytical expressions for the dispersion of the maximum likelihood estimates (α_0^*, β_0^*). It is best to use the analytical method which examines an ensemble of received oscillations $x(t)$ with fixed signal parameter value.

It should be noted that dispersion is not the only possible scattering characteristic of a random quantity. It is often convenient to use a different characteristic. In the case of a unidimensional random quantity α_0^* the concentration or scattering can be characterized by the scattering interval ($\alpha_0+\sqrt{3}\delta_\alpha$, $\alpha_0-\sqrt{3}\delta_\alpha$). If we replace the true distribution α_0^* by a uniform distribution over the scattering interval, the value of the first and second moment of the distribution remains the same. Analogously, in case of a two dimensional random quantity (α_0^*, β_0^*), its concentration about the "center of gravity" of α_0, β_0 is characterized by the scattering ellipse

$$\frac{1}{1-\rho^2}\left[\frac{(\alpha_0^*-\alpha_0)^2}{\delta_\alpha^2}-\frac{2\rho(\alpha_0^*-\alpha_0)(\beta_0^*-\beta_0)}{\delta_\alpha\delta_\beta}+\frac{(\beta_0^*-\beta_0)^2}{\delta_\beta^2}\right]=4, \tag{4.39}$$

where ρ is the correlation coefficient

$$\rho=\frac{\delta^2_{\alpha\beta}}{\delta_\alpha\delta_\alpha};\qquad\qquad(4.40)$$

$$\alpha_0=\langle\alpha_0^*\rangle,\ \beta_0=\langle\beta_0^*\rangle,$$

and δ^2_α, δ^2_β, $\delta^2_{\alpha\beta}$ are the second central moments,

$$\delta^2_\alpha=\langle(\alpha_0^*-\alpha_0)^2\rangle,\quad \delta^2_\beta=\langle(\beta_0^*-\beta_0)^2\rangle,\qquad(4.41)$$

$$\delta^2_{\alpha\beta}=\langle(\alpha_0^*-\alpha_0)(\beta_0^*-\beta_0)\rangle.\qquad(4.42)$$

If we replace the distribution of the random quantity (α_0^*, β_0^*) by a uniform distribution in the area bounded by the scattering ellipse, the value of the first and second moments of the distribution remain unchanged. Obviously, assignment of the matrix of second moments defines the scattering ellipse, and conversely. It is therefore desirable to have analytical formulas for the dispersions δ^2_α, δ^2_β as well as for the second mixed central moment $\delta^2_{\alpha\beta}$.

We first assume that there is only one useful parameter α. Then, according to the above definition of the ensemble x(t), it should be assumed that

$$x(t)=n(t)+\varepsilon_0 s(t,\alpha_0,\varphi_0),\qquad(4.43)$$

where α_0, ϕ_0, ε_0 are fixed values of the signal parameters. The a posteriori probability function, or its equivalent in the sense of information, the output $Y(\alpha)$, is random in this case and can be represented as the sum of the mathematical expectation and a random function with null mean

$$Y(\alpha)=\langle Y(\alpha)\rangle+Y^0(\alpha).\qquad(4.44)$$

Here and below the superscript zero indicates deviation of the function from the mathematical expectation

$$Y^0(\alpha) = Y(\alpha) - \langle Y(\alpha) \rangle. \tag{4.45}$$

In accordance with the maximum likelihood method the estimate α_0^* is found from the equation

$$\frac{d}{d\alpha} \langle Y(\alpha_0^*) \rangle + \frac{d}{d\alpha} Y^0(\alpha_0) = 0. \tag{4.46}$$

In (4.46) the conventional notation for the derivatives has the usual meaning

$$\frac{d}{d\alpha} Y(\alpha_0) = Y'(\alpha_0) = \left[\frac{d}{d\alpha} Y(\alpha) \right]_{\alpha=\alpha_0},$$

and replacement of the argument α_0^* with α^* in the random quantity $\frac{d}{d\alpha} Y^0(\alpha)$ is justified by the practical equality of the statistical characteristics of the random process $Y^0(\alpha)$ at points α_0^* and α_0 which are close together. In addition, the process $Y^0(\alpha)$ can be assumed stationary in all of the practical applications examined below.

Equation (4.46) makes it possible in principle to represent the estimate α_0^*, which is random in nature, as a function of fixed values of the parameters α_0, ε_0 and the random quantity $\frac{d}{d\alpha} Y^0(\alpha_0)$ with known statistical characteristics, after which the dispersion of the estimate α_0^* can be found.

In the case of very strong signals the error of the optimal system $(\alpha_0^* - \alpha_0)$ can be assumed small, and its mathematical expectation can be assumed to be zero. In this case likelihood equation (4.46) takes on the form

$$(\alpha_0^* - \alpha_0)\frac{d^2}{d\alpha^2}\langle Y(\alpha_0)\rangle + \frac{d}{d\alpha}Y^0(\alpha_0) = 0, \tag{4.47}$$

after which we obtain this expression for the potential dispersion of the estimates*

$$\delta_\alpha^2 = \langle(\alpha_0^* - \alpha_0)^2\rangle = \frac{\left\langle\left[\frac{d}{d\alpha}Y^0(\alpha_0)\right]^2\right\rangle}{\left[\frac{d^2}{d\alpha^2}\langle Y(\alpha_0)\rangle\right]^2}. \tag{4.48}$$

When there are two useful parameters α and β it is assumed analogously that the received oscillation contains a signal with fixed parameters α_0, β_0, ε_0, ϕ_0. The maximally reliable estimate (α_0^*, β_0^*) is found from the optimum output $Y(\alpha,\beta)$ by solving the system of likelihood equations

$$\begin{aligned} &\frac{\partial}{\partial\alpha}\langle Y(\alpha_0^*, \beta_0^*)\rangle + \frac{\partial}{\partial\alpha}Y^0(\alpha_0, \beta_0) = 0, \\ &\frac{\partial}{\partial\beta}\langle Y(\alpha_0^*, \beta_0^*)\rangle + \frac{\partial}{\partial\beta}Y^0(\alpha_0, \beta_0) = 0. \end{aligned} \tag{4.49}$$

The solution of system (4.49) makes it possible in principle to find expressions for the estimates α_0^* and β_0^* in the form of a function of the random quantities $\frac{\partial}{\partial\alpha}Y^0(\alpha_0, \beta_0)$, $\frac{\partial}{\partial\beta}Y^0(\alpha_0, \beta_0)$ and the true values of the parameters α_0, β_0, ε_0, after which the scattering characteristics of the two dimensional random quantity can be determined.

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* Here and below the second central moments of the estimate for fixed values of the signal parameters are designated by the symbols δ_α^2, δ_β^2, $\delta_{\alpha\beta}^2$ without the subscript su.

In the case of very strong signals, system of equations (4.49) can be written approximately as follows:

$$(\alpha_0^* - \alpha_0)\frac{\partial^2}{\partial\alpha^2}\langle Y(\alpha_0,\beta_0)\rangle + (\beta_0^* - \beta)\frac{\partial^2}{\partial\alpha\partial\beta}\langle Y(\alpha_0,\beta_0)\rangle + \frac{\partial}{\partial\alpha}Y^0(\alpha_0,\beta_0) = 0,$$

$$(\alpha_0^* - \alpha_0)\frac{\partial^2}{\partial\alpha\partial\beta}\langle Y(\alpha_0,\beta_0)\rangle + \frac{\partial^2}{\partial\beta^2}\langle Y(\alpha_0,\beta_0)\rangle + \frac{\partial}{\partial\beta}Y^0(\alpha_0,\beta_0) = 0. \quad (4.50)$$

We then obtain in explicit form the following expressions for the deviation of the estimates:

$$(\alpha_0^* - \alpha_0) = \frac{\frac{\partial^2}{\partial\beta^2}\langle Y(\alpha_0,\beta_0)\rangle \frac{\partial}{\partial\alpha}Y^0(\alpha_0,\beta_0) - \frac{\partial^2}{\partial\alpha\partial\beta}\langle Y(\alpha_0,\beta_0)\rangle \frac{\partial}{\partial\beta}Y^0(\alpha_0,\beta_0)}{\frac{\partial^2}{\partial\alpha^2}\langle Y(\alpha_0,\beta_0)\rangle \frac{\partial^2}{\partial\beta^2}\langle Y(\alpha_0,\beta_0)\rangle - \left[\frac{\partial^2}{\partial\alpha\partial\beta}\langle Y(\alpha_0,\beta_0)\rangle\right]^2},$$

$$(\beta_0^* - \beta_0) = \frac{\frac{\partial^2}{\partial\alpha^2}\langle Y(\alpha_0,\beta_0)\rangle \frac{\partial}{\partial\beta}Y^0(\alpha_0,\beta_0) - \frac{\partial^2}{\partial\alpha\partial\beta}\langle Y(\alpha_0,\beta_0)\rangle \frac{\partial}{\partial\alpha}Y^0(\alpha_0,\beta_0)}{\frac{\partial^2}{\partial\alpha^2}\langle Y(\alpha_0,\beta_0)\rangle \frac{\partial^2}{\partial\beta^2}\langle Y(\alpha_0,\beta_0)\rangle - \left[\frac{\partial^2}{\partial\alpha\partial\beta}\langle Y(\alpha_0,\beta_0)\rangle\right]^2}. \quad (4.51)$$

In nearly all calculations of the potential accuracy of the signal parameter estimate presented below, the optimum output $Y(\alpha,\beta)$ for a signal with fixed parameters α_0, β_0, ε_0, ϕ_0, in the vicinity of the estimate α_0^*, β_0^* can be represented in the form

$$Y(\alpha,\beta) = \varepsilon_0 a^2 \Psi(\alpha - \alpha_0, \beta - \beta_0) + a\theta(\alpha,\beta), \quad (4.52)$$

where the function $\Psi(\alpha,\beta)$ is determined by means of (4.33), with a some number, and $\theta(\alpha,\beta)$ a normal and normalized random process such that

$$\langle\theta(\alpha_1,\beta_1)\theta(\alpha_2,\beta_2)\rangle = \Psi(\alpha_1 - \alpha_2, \beta_1 - \beta_2). \quad (4.53)$$

The following relationships followed directly from (4.53):

$$\left\langle\left[\frac{\partial}{\partial\alpha}\vartheta(\alpha,\beta)\right]^2\right\rangle=-\left[\frac{\partial^2}{\partial\alpha^2}\Psi(\alpha,0)\right]_{\alpha=0}\equiv-\Psi''_{\alpha}(0,0), \tag{4.54}$$

$$\left\langle\left[\frac{\partial}{\partial\beta}\vartheta(\alpha,\beta)\right]^2\right\rangle=-\left[\frac{\partial^2}{\partial\beta^2}\Psi(0,\beta)\right]_{\beta=0}\equiv-\Psi''_{\beta}(0,0), \tag{4.55}$$

$$\left\langle\frac{\partial}{\partial\alpha}\vartheta(\alpha,\beta)\frac{\partial}{\partial\beta}\vartheta(\alpha,\beta)\right\rangle=-\left[\frac{\partial^2}{\partial\alpha\partial\beta}\Psi(\alpha,\beta)\right]_{\substack{\alpha=0\\ \beta=0}}\equiv-\Psi''_{\alpha\beta}(0,0), \tag{4.56}$$

By substituting (4.52) and (4.54)-(4.56) in (4.51), after averaging we obtain these comparatively simple analytical formulas

$$\delta_{\alpha}^2=\langle(\alpha_0^*-\alpha_0)^2\rangle=\frac{1}{\epsilon_0^2a^2|\Psi''_{\alpha}(0,0)|}\,\frac{1}{1-r^2}, \tag{4.57}$$

$$\delta_{\beta}^2=\langle(\beta_0^*-\beta_0)^2\rangle=\frac{1}{\epsilon_0^2a^2|\Psi''_{\beta}(0,0)|}\,\frac{1}{1-r^2}, \tag{4.58}$$

$$\delta_{\alpha\beta}^2=\langle(\alpha_0^*-\alpha_0)(\beta_0^*-\beta_0)\rangle=\frac{1}{\epsilon_0^2a^2\Psi''_{\alpha\beta}(0,0)}\,\frac{r^2}{1-r^2}, \tag{4.59}$$

where

$$r^2=\frac{[\Psi''_{\alpha\beta}(0,0)]^2}{\Psi''_{\alpha}(0,0)\Psi''_{\beta}(0,0)}. \tag{4.60}$$

Analogously, when there is a single useful parameter α the optimum output can be represented as

$$Y(\alpha)=\epsilon_0a^2\Psi(\alpha-\alpha_0)+a\vartheta(\alpha) \tag{4.61}$$

and analytical expression (4.48) can be transformed as follows:

$$\delta_{\alpha}^2=\langle(\alpha_0^*-\alpha_0)^2\rangle=\frac{1}{\epsilon_0^2a^2|\Psi''_{\alpha}(0)|}. \tag{4.62}$$

When receiving signals with known intensity the parameter ε_0 in all of the formulas should be set equal to 1.

The mean values $\langle\delta_\alpha^2\rangle$, $\langle\delta_\beta^2\rangle$ and $\langle\delta_{\alpha\beta}^2\rangle$ of the obtained scattering characteristics δ_α^2, δ_β^2, and $\delta_{\alpha\beta}^2$ are determined by statistical averaging of the latter with respect to all possible values of the signal parameters α_0, β_0, ε_0, ϕ_0. When the optimum output $Y(\alpha,\beta)$ can be represented in form (4.52), the characteristics δ_α^2, δ_β^2, $\delta_{\alpha\beta}^2$, like the characteristics $[\delta_\alpha^2]_x$, $[\delta_\beta^2]_x$ depend only upon the intensity parameter when condition (4.33) is satisfied. Furthermore,

$$\langle\delta_\alpha^2\rangle=\int_0^\infty \delta_\alpha^2\, p(\varepsilon_0)\,d\varepsilon_0,\quad \langle\delta_\beta^2\rangle=\int_0^\infty \delta_\beta^2\, p(\varepsilon_0)\,d\varepsilon_0, \tag{4.63}$$

$$\langle\delta_{\alpha\beta}^2\rangle=\int_0^\infty \delta_{\alpha\beta}^2\, p(\varepsilon_0)\,d\varepsilon_0$$

with an output with the form (4.52) and known intensity of the received signals, when $p(\varepsilon)=\delta(\varepsilon-1)$, equality (4.5) holds

$$\langle\delta_\alpha^2\rangle=\delta_\alpha^2,\ \langle\delta_\alpha^2\rangle=\delta_\beta^2,\ \langle\delta_{\alpha\beta}^2\rangle=\delta_{\alpha\beta}^2.$$

The method presented in this section for analyzing the error dispersion can also be used when the output $Y(\alpha,\beta)$ is non-optimal. This capability will be utilized in Chapter 8.

4.4. Method for Constructing Optimal Circuits for Obtaining Estimate and Error Signal

A direct method for obtaining the best estimate is to form the optimum output and find its highest point. In many cases, especially when the output is a function of several useful parameters, this method for obtaining the estimate encounters practical difficulties. It is desirable, using some of the useful parameters, such as the parameter α, to obtain an estimate which is as close as possible to the best estimate without forming and reproducing the output for all values of the parameter α in the interval $(\alpha_\Phi-\alpha_m,\ \alpha_\Phi+\alpha_m)$ of the possible values of that parameter. It is assumed that the fact of the occurrence of a signal with parameter α belonging to the aforementioned interval is established reliably and is not subject to doubt, and that α_Φ is the position of some fixed point of the system (the temporal location of the gating pulses, the angular position of the axis of the antenna system, etc.).

An analogous situation occurs for the error signal, whose derivation is a rougher result of radar observation than the obtaining of a parameter estimate. The error signal d is usually defined as a number which must be proportional to the deviation of the fixing point α_Φ from α_0 -- the true value of the parameter α. When the reception is done against the background of interference, the true value of the parameter cannot be found nor, accordingly, can the deviation from that value be determined. The potential capabilities of the system are exhausted by the capability of forming an error signal d which is proportional to the deviation of the fixing point α_Φ from the best estimate, in the capacity of which can serve the maximum likelihood estimate α_0^*

$$d = k(\alpha_\Phi - \alpha_0^*). \tag{4.64}$$

We shall assume that the proportionality coefficient k in (4.64) is a positive random quantity. If the number k is known a priori, obtaining the error signal is obviously a problem which is identical to the problem of obtaining the estimate.

We note that in the general case the error signal on the interval $(\alpha_\Phi-\alpha_m, \alpha_\Phi+\alpha_m)$ can be expressed through an arbitrary monotonic function ϕ of the deviation $\alpha_\Phi-\alpha_0^*$

$$\tilde{d} = \varphi[k(\alpha_\Phi - \alpha_0^*)]. \tag{4.65}$$

Assumption (4.64) instead of (4.65) causes no loss in generality, since the error signal d can be obtained from $\tilde{d}$ by employing the appropriate one-to-one transformation.

The present section presents an analytical method for defining optimal schemes for obtaining the estimate and error signal based on employing and solving a system of equations of the type (4.38).

We shall assume that the signal contains two useful parameters α and β. The optimum output $Y(\alpha,\beta)$, as follows from (4.29) and (4.33), can be represented as the series (4.20)

$$Y(\alpha,\beta)=\varepsilon_0^*\left|\int_{-\infty}^{\infty}\overline{S(t,\alpha,\beta)}S(t,\alpha_0^*,\beta_0^*)dt\right|= \tag{4.66}$$
$$=\varepsilon_0^*\sum_{ij}a_{ij}(\alpha-\alpha_0^*)^i(\beta-\beta_0^*)^j;\ (j+i=2n;i,j,n=0,1,2,\ldots),$$

where

$$a_{ij}=\frac{1}{(i+j)!}\left[\frac{\partial^{i+j}}{\partial\alpha^i\partial\beta^j}\left|\int_{-\infty}^{\infty}\overline{S(t,\alpha,\beta)}S(t,\alpha_0^*,\beta_0^*)dt\right|\right]_{\substack{\alpha=\alpha_0^*\\ \beta=\beta_0^*}} \tag{4.67}$$

The use of assumption (4.33) simplifies the illustration of the method, but is not obligatory.

We shall assume further that the estimate β_0^* is obtained in some way, and that it is required to find the best estimate of the parameter α without forming an output for all values of that parameter. In accordance with the above, one method for obtaining the best estimate of one parameter is to solve a system of two equations. The choice of the system of equations for obtaining the estimate can be made in various ways. We shall limit ourselves to the class of equations in which one of the equations defines the error signal (i.e., expresses the error as a function of the estimate), and the other makes it possible to define or exclude the unknown coefficient from the first equation. For example, we can use the following as our system of equations:

i.e., formation of the optimum error signal consists of multiplying the received oscillation x(t) by the function $з'_{\alpha}(t,\alpha_{\Phi})$ and integrating the product obtained. In order to form the estimate it is necessary to form a second number $\mathfrak{D}$ or compose a second equation

$$\mathfrak{D}=Y''_{\alpha}(\alpha_{\Phi})=\int_{-\infty}^{\infty} x(t)з''_{\alpha}(t,\alpha_{\Phi})\,dt \tag{4.71}$$

by performing analogous operations with the function $з''_{\alpha}(t,\alpha_{\Phi})$.

The problem of determining an optimum scheme, as well as the problem of composing a system of equations for obtaining the estimate, does not have a single solution. Another possibility for obtaining the optimum error signal and best estimate is, for example, to form two values of the output $Y(\alpha_{\Phi}+\Delta,\beta_0^*)$ and $Y(\alpha_{\Phi}-\Delta,\beta_0^*)$ for the points $\alpha_{\Phi}+\Delta,\beta_0^*$ and $\alpha_{\Phi}-\Delta,\beta_0^*$ offset symmetrically with respect to the fixing point. The optimum error signal is then obtained by composing the difference

$$d=Y(\alpha_{\Phi}+\Delta,\beta_0^*)-Y(\alpha_{\Phi}-\Delta,\beta_0^*)\approx 2\Delta Y'_{\alpha}(\alpha_{\Phi},\beta_0^*), \tag{4.72}$$

while an additional equation for obtaining the estimate can be the sum

$$\mathfrak{D}=Y(\alpha_{\Phi}+\Delta,\beta_0^*)+Y(\alpha_{\Phi}-\Delta,\beta_0^*). \tag{4.73}$$

Keeping (4.66) in mind, we obtain approximately

$$\alpha_0^*=\alpha_{\Phi}-\frac{Y(\alpha_{\Phi}+\Delta,\beta_0^*)-Y(\alpha_{\Phi}-\Delta,\beta_0^*)}{Y(\alpha_{\Phi}+\Delta,\beta_0^*)+Y(\alpha_{\Phi}-\Delta,\beta_0^*)}\cdot\frac{a_{00}}{2\Delta a_{20}}. \tag{4.74}$$

Consequently, this method for designing optimal systems for obtaining the signal parameter estimate consists of the following. Based on representing

the optimum output as a function of the maximum likelihood estimate (4.27)-(4.39) a system of equations is composed whose solution is the estimate. The number of equations n must exceed the number of measured parameters by one. When obtaining the estimate of one parameter, the error signal as a function of the estimate can be employed as one of the equations. Then, returning to the representation of the optimum output as a function of the received oscillation x(t), the equations constructed are modeled with the help of an electrical circuit. The circuit forms n numbers, which define the value of the free terms in the system of equations and, consequently, its solution. The numbers formed by the circuit are input to the decision device. The algorithm for obtaining the estimate in the decision device is determined directly by the system of equations selected, and may be very simple, especially if the errors are small and representation (4.35) can be used. Practical examples of the application of this method for constructing optimum circuits for obtaining estimates and optimum error signal formation circuits will be presented during the investigation of direction finding systems.

4.5. Potential Sampling Accuracy of Range Coordinate from Stationary Targets

As our first application of the theory of radar signal parameter estimates we shall calculate the potential accuracy of sampling the delay of a signal reflected from a stationary target. The only useful parameter is the delay time τ. The modulation function $S(t-\tau)$ in this case satisfies condition (4.33). We shall assume at the outset that the parasitic random parameters ϕ and ε are absent. Then the optimum output

$$Y(\tau)=\frac{2}{N_0}\int_{-\infty}^{\infty} x(t)\,s(t-\tau)\,dt \qquad (4.75)$$

is the high frequency function with slowly varying envelope $L[Y(\tau)]$. Determination of the maximum point of function $Y(\tau)$ by solving the likelihood equation

$$Y'(\tau)=0,\ (\tau_0-\varepsilon<\tau<\tau_0+\varepsilon) \qquad (4.76)$$

yields an ambiguous result because of the high frequency carrier. The region $(\tau_0-\varepsilon,\ \tau_0+\varepsilon)$ in this case designates the vicinity of the point τ_0 -- the true delay time of the reflected signal. Conversely, we can assume for

the envelope of the output $L[Y(\tau)]$ that in the region $(\tau_0-\varepsilon,\ \tau_0+\varepsilon)$ the probability of the occurrence of more than one maximum is near zero. For this reason, we shall initially seek the maximum point of the function $Y(\tau)$ from the envelope of $L[Y(\tau)]$, i.e., from the equation

$$\frac{d}{d\tau} L[Y(\tau)] = 0, \quad (\tau_0 - \varepsilon < \tau < \tau_0 + \varepsilon), \tag{4.77}$$

after which it will be possible to answer the question as to whether the high frequency carrier can be utilized.

Assuming that

$$x(t) = n(t) + s(t - \tau_0) \tag{4.78}$$

the expression for the envelope of $L[Y(\tau)]$, assuming that the signal is significantly stronger than the interference, can be reduced to the form of (4.61) on the basis of (1.98)

$$\begin{aligned} L[Y(\tau)] &= 2q^2\Psi(\tau - \tau_0) + \sqrt{2}q\theta(\tau) \approx \\ &\approx 2q^2\left[1 - \frac{1}{2}\beta_\tau^2(\tau - \tau_0)^2\right] + \sqrt{2}q\,\theta(\tau), \end{aligned} \tag{4.79}$$

where $\Psi(\tau)$ is the correlation function of the modulation, β_τ is a parameter which determines the signal spectrum width by means of formula (1.23), and $\theta(\tau)$ is a normal random process with correlation function

$$\langle\theta(\tau_1)\,\theta(\tau_2)\rangle = \Psi(\tau_1 - \tau_2). \tag{4.80}$$

The solution of likelihood equation (4.77) is determined by formula (4.62)

$$\delta_\tau^2 = \langle(\tau_0^* - \tau_0)^2\rangle = \frac{1}{2q^2\beta_\tau^2} \tag{4.81}$$

or

$$\delta_\tau^2 = \frac{N_0}{\int\limits_{-\infty}^{\infty} |S'(t)|^2 dt}. \tag{4.82}$$

Formula (4.81) was derived by Woodward [8] by calculating the dispersion of the averaged a posteriori distribution.

If the mean square error δ_τ is significantly smaller than the period of the high frequency oscillations of the signal

$$\frac{1}{\sqrt{2}q\beta_\tau} \ll \frac{1}{f_0}, \tag{4.83}$$

knowledge of the envelope estimate makes it possible to reduce the extent of the vicinity of the point τ_0 in initial equation (4.76) enough to eliminate the ambiguity in the solution. In this case, solving initial equation (4.76) under condition (4.78) yields

$$\delta_\tau^2 = \frac{1}{2q^2(2\pi f_0)^2}. \tag{4.84}$$

Consequently, if condition (4.83) is satisfied, i.e., if the modulation principle and power ratio q^2 are such that the envelope coordinate sample eliminates the ambiguity, the high frequency carrier of the signal can be used to sample the range. The potential sampling accuracy is then expressed by formula (4.84). Condition (4.83) can be satisfied for certain radio navigation and remote control systems. The reverse inequality

$$\frac{1}{\sqrt{2}q\beta_\tau} \gg \frac{1}{f_0} \tag{4.85}$$

holds for radar systems, and the potential range sampling accuracy is determined by formula (4.81) or (4.82).

Thus, even if the initial phase of the received signals is not a statistically independent random parameter, and contains useful information, this information cannot be used in real radar lines to increase the accuracy of the range coordinate sampling, or to increase detection reliability. Therefore, we shall assume everywhere below that the initial phase of the received signals is a random quantity which is distributed uniformly over the interval 0, 2π. This assumption also corresponds to actual operating conditions of radar systems.

The potential range coordinate sampling accuracy in radar systems, as follows from the above formulas, depends in the final analysis only upon the spectral intensity of the interference n_0 and the total "energy" $\int_{-\infty}^{\infty}|S'(t)|^2dt$, contained in the derivative of the complex envelope of the signal. The accuracy is determined by the parameter β_τ when the signal/noise ratio is given. Therefore, in contrast to the threshold ratio, range coordinate sampling accuracy depends significantly upon the form of the signals and is the higher, the wider the modulation spectrum and the more of the modulating signal energy is concentrated in the region of high modulation frequencies.

We can pose the problem of determining the form of the signal which provides the highest sampling accuracy for the parameter τ for a given signal/noise power ratio q^2 and assuming that the signal spectrum stays practically within the band $(f_0-F_m,\ f_0+F_m)$. Such a signal is

$$s(t)=1(t,\ T)S_0\cos 2\pi F_m t\cos 2\pi f_0 t, \tag{4.86}$$

since its spectrum approximates the form $\delta(f-f_0+F_m)+\delta(f-f_0-F_m)$ and has the maximum possible "dispersion" $(2\pi F_m)^2$ with respect to $f=f_0$.

However, no matter how high the frequencies F_m function (4.86) is not suitable for use in the capacity of radar signals.

Some limitation of the area of application of formula (4.81) must also be emphasized. This formula is valid only for signals above the threshold. As the parameter β_τ increases the threshold increases slowly, but monotonically. Therefore, for a given interference level N_0 it is not possible to increase the sampling accuracy without limit by changing the form of the signal without changing its power. Sampling accuracy can

be increased in this way only as long as the reliable detection condition $q^2 \geq q^2_{thr}$ is retained. In addition, formulas (4.81) and (4.82) are based on representation (1.24) and, consequently, are valid only for those signals and those intervals τ for which the approximate equality

$$\Psi(\tau) \approx 1 - \frac{1}{2}\beta_\tau^2\tau^2.$$

holds.

This representation is not applicable, for example, to the frequently employed idealization of the envelope of valid signals in the form of square pulses with correlation function

$$\Psi(\tau) = \begin{cases} 1 - \frac{|\tau|}{\tau_H}, & |\tau| \leq \tau_H, \\ 0, & |\tau| \geq \tau_H. \end{cases} \tag{4.87}$$

In this case, using formula (4.17) for the mean square errors and representation (4.28), assuming $\varepsilon=\varepsilon_0^*=1$, we obtain

$$\delta_\tau^2 = k_x \int_{\tau_0^*-\tau_H}^{\tau_0^*+\tau_H} (\tau_0^* - \tau)^2 I_0\left[2q^2\left(1 - \frac{|\tau_0^* - \tau|}{\tau_H}\right)\right] d\tau \approx$$

$$\approx \frac{\int_0^{\tau_H} x^2 \exp\left(-\frac{2q^2x}{\tau_H}\right) dx}{\int_0^{\tau_H} \exp\left(-\frac{2q^2x}{\tau_H}\right) dx} \tag{4.88}$$

or

$$\delta_\tau^2 \approx \frac{\tau_H^2}{2q^4} = \frac{2N_0^2}{S_0^4},$$

where S_0 is the amplitude of the input signal pulse. According to the above, for the signals in question the dispersion (4.88), like the dispersion (4.81), is the same as the averaged dispersion of the estimates $\langle\delta_\tau^2\rangle$.

The potential sampling accuracy for square pulsed signals depends only upon the spectral intensity of the interference and the signal amplitude, but not upon the pulse duration. Expression (4.88) can be viewed as the limit which the potential sampling accuracy approaches as the slope of the leading edge of the signal pulses increases without limit.

In order to obtain the final expressions for the sampling accuracy of the range coordinate of stationary targets, we must also allow for the occurrence of two parasitic parameters in the received signal: the random initial phase ϕ and the random intensity ε. In this case, the optimal output $Y(\tau)$ can be the envelope of oscillation (4.75). Assuming

$$x(t)=n(t)+\varepsilon_0 s(t-\tau_0, \varphi) \tag{4.89}$$

and performing transformations analogous to those which lead to (4.79), we obtain

$$Y(\tau)=\varepsilon_0 2q^2\Psi(\tau-\tau_0)+\sqrt{2}q\theta(\tau) \tag{4.90}$$

and

$$\delta_\tau^2=\frac{1}{2q^2\varepsilon_0^2\beta_\tau^2}. \tag{4.91}$$

As might be expected, in accordance with the conclusions above the occurrence of the random initial phase of the received signal does not increase the dispersion of the samples of the range coordinate.

As regards the influence of random intensity, comparison of formulas (4.91) and (4.81) permits the conclusion that the potential dispersion of the samples does not depend upon whether or not the intensity of the received signal is known, but rather upon the intensity $\varepsilon_0{}^2Q^2$ of the signal which is actually received.

The averaged potential dispersion of the errors $\langle \delta_\tau^2 \rangle$ is obtained theoretically by statistically averaging with respect to the parameter ε_0 the dispersion δ_τ^2 obtained assuming that the parameter ε_0 is fixed. However, averaging cannot be done directly with respect to all possible

values of ε_0 of the obtained quantity (4.91), since all of our formulas as based on the assumption that reliable detection occurs, and are consequently valid only for sufficiently large ε_0. In addition, formula (4.91) has a perfectly clear physical meaning -- it represents the potential accuracy for received signal energy of $\varepsilon_0{}^2Q^2$, regardless of whether or not the intensity of the received signal is known while the estimate is being obtained. This formula provides a sufficiently good characterization of the potential capabilities of the system. Therefore, we shall not concern ourselves here and below with seeking intelligent methods of averaging the expressions for the dispersion of the estimates of various parameters with respect to ε_0.

During reception against a background of random interference with spectral intensity depending upon frequency (normal correlated interference), the analytical expression for potential accuracy (4.91) takes on the form

$$\delta_\tau^2 = \frac{N_0}{2Q^2_{\text{ЭКВ}}\beta^2_{\tau\,\text{ЭКВ}}\varepsilon_0^2}. \tag{4.92}$$

The parameters $Q^2_{\text{ЭКВ}}$ and $\beta^2_{\tau\,\text{ЭКВ}}$ were defined in § 3.10.* N_0 is the spectral intensity of the interference at carrier frequency f_0.

Chapter 7. Azimuth Scanning and Target Direction Finding

7.1. Azimuth Scanning by Flat Rotation of Directivity Pattern

It has been assumed up to now that the scanning has been done in one fixed direction. We shall now assume that scanning is also done by angular coordinate θ within some arbitrary sector $(\theta_{min}, \theta_{max})$, which we shall assume for definition to be equal to $(0, \theta_m)$. We shall begin by studying the basic scanning method -- that of flat rotation (or rocking) of the pattern of the antenna system within the working sector $(0, \theta_m)$. The (voltage) directivity patterns of the transmitting and receiving antennas will be assumed to be real functions of θ and will be designated $\gamma_I(\theta)$ and $\gamma_{II}(\theta)$. The direction of maximum gain of both antennas corresponds to the angle $\theta=0$.

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*[Chapter 3 not included in this translation. Tr.]

The objective of the investigation is to obtain analytical formulas for the threshold signal and for the potential accuracy of the estimate of the angular coordinate, which is sampled jointly with the range coordinate. In order to do this, we must write the analytical expression for the optimum output and study its statistical characteristics in the vicinity of the estimate (in the region of high correlation) in a manner analogous to that employed in Chapter 5 for signals containing the two useful parameters τ and Φ. The problem of determining the optimum azimuth search mode is not dealt with. It is assumed that the trials are made at fixed intervals (the signal has fixed duration). It is assumed accordingly that the angular scanning rate or rotation rate of the directivity pattern is a constant quantity equal to ω deg/sec, and that ωt is the current angle of revolution (angular position) of the pattern. We shall examine a single scan cycle, so that the angular position of the target

$$\theta = \omega t_A, \tag{7.1}$$

where t_A is the instant in time or the time offset corresponding to rotation of the antenna system through an angle equal to the angular coordinate of the target. Then the function

$$s(t)\,\gamma_1[\omega(t_A - t)]$$

represents the signal emitted in the direction θ, and the received signal which is reflected from the target and whose coordinates and velocity correspond to the parameters τ, t_A and Φ, will be expressed by the function

$$\varepsilon\gamma_1[\omega(t_A + \tau - t)]\,f_{11}[\omega(t_A - t)]\,s(t - \tau,\ \varphi,\ \Phi). \tag{7.2}$$

In writing (7.2) it is assumed in accordance with the actual operating conditions of radar systems that the target irradiation time in each space scanning cycle does not exceed the coherence integral, and that the fluctuation in the scanning rate ω caused by movement of the target is negligibly small.

In theoretical studies it is convenient to approximate the directivity pattern of radar antennas by the bell-shaped function

$$\gamma_{I}(\theta)=\frac{k}{\sqrt{\Delta_{I}}}\exp\left(-2\ln 2\frac{\theta^{2}}{\Delta_{I}^{2}}\right),$$

$$\gamma_{II}(\theta)=\frac{k}{\sqrt{\Delta_{II}}}\exp\left(-2\ln 2\frac{\theta^{2}}{\Delta_{II}^{2}}\right). \tag{7.3}$$

Here Δ_I and Δ_{II} are the respective pattern widths of the transmitting and receiving antennas taken at the half-power level, and k is the proportionality coefficient.

Then the generalized directivity pattern of the transmitting and receiving antennas also represents a bell-shaped function

$$\gamma_{I}[\omega(t_{A}+\tau-t)]\gamma_{II}[\omega(t_{A}-t)=$$
$$=\frac{k^{2}}{\sqrt{\Delta_{I}\Delta_{II}}}\exp\left[-2\ln 2\frac{(\omega\tau)^{2}}{\Delta_{I}^{2}+\Delta_{II}^{2}}\right]\exp\times$$
$$\times\left[-2\ln 2\frac{\omega^{2}(\Delta_{I}^{2}+\Delta_{II}^{2})}{\Delta_{I}^{2}\Delta_{II}^{2}}\left(t_{A}+\frac{\Delta_{II}^{2}}{\Delta_{I}^{2}+\Delta_{II}^{2}}\tau-t\right)^{2}\right] \tag{7.4}$$

with pattern width

$$\Delta_{\Sigma}=\frac{\Delta_{I}\Delta_{II}}{\sqrt{\Delta_{I}^{2}+\Delta_{II}^{2}}}, \tag{7.5}$$

which corresponds to an effective signal duration of

$$T = \frac{1}{2}\sqrt{\frac{\pi}{\ln 2}}\frac{\Delta_\Sigma}{\omega}. \tag{7.6}$$

The occurrence of a time offset between reception and transmission results in an offset of

$$\nu\tau = \frac{\Delta_{II}^2}{\Delta_I^2 + \Delta_{II}^2}\tau \tag{7.7}$$

between the moment at which the maximum of the pattern is directed toward the target and the moment corresponding to the maximum amplitude of the received signal, as well as a reduction in the resultant gain of the antenna system, which is expressed by the first exponential factor in (7.4).

In order to make it easier to write subsequent formulas, we shall assume that the time offset $\nu\tau$ in (7.4) is small and can be disregarded. Then the reflected signal takes on the form

$$\varepsilon s(t-\tau,\ \varphi,\ \Phi,\ \theta) = \varepsilon\gamma\left[\omega\left(t_A - t\right)\right]\mathrm{Re}\left\{S(t-\tau)\,e^{j[2\pi f_0(t-\tau)+2\pi\Phi t+\varphi]}\right\}, \tag{7.8}$$

where $\gamma(\theta)$ is the generalized directivity pattern of the transmitting and receiving antennas. The simplification in writing (7.8) which comes about by disregarding the quantity $\nu\tau$ is acceptable in most cases. In the more general case the time offset $\nu\tau$ can be included in the parameter t_A, which again leads to (7.8).

The reflected signal is structurally the same as the signals studied in the preceding chapters, except for the fact that the time position of the function γ, which plays the role of cutoff function, depends upon the azimuth of the target. For this reason, further analysis duplicates to a significant extent the analysis presented in Chapters 5 and 6. We limit ourselves here to reproducing the order of the calculation, dealing in detail only with singular features associated with the occurrence of the additional random parameters -- the delay time of the cutoff function.

The signal or modulating function S(t) figuring in (7.8) can be any of the radar signals examined above, including the sum of some number of incoherent elementary signals. If S(t) is a coherent or incoherent sequence of periodically repeating signals S_{TM} (t), the modulating function in (7.8) will appear as

$$S(t)=\sum_{i=-\infty}^{\infty} S_{T_M}(t-kT_M-\xi)e^{j\varphi_k}, \tag{7.9}$$

where ξ is a quantity uniformly distributed over the interval $|\xi| \leq 0.5T_M$ and which allows for the absence of synchronization between the rotation of the directivity pattern and the modulation. However, ξ is a parameter of the emitted signal which can be allowed for exactly during the processing, and therefore must be viewed as a known non-random parameter. In addition, for coherent signals all of the ϕ_k in (7.9) must be assumed to be equal non-random quantities.

The likelihood coefficient for the parameters τ,θ and Φ, which determines the potential capabilities of the system and the optimum processing circuit, is expressed by the same formulas which were derived in our examination of spatial scanning in one fixed direction. The only difference is that the function Y which figures in the formulas for the likelihood coefficient presented in Chapters 5 and 6 are now in the general case functions of three parameters and have the following appearance:

-- for coherent processing [formula (5.12)]

$$Y(\tau, \theta, \Phi)=L\left\{\frac{2}{N_0}\int_{-\infty}^{\infty} x(t)\gamma[\omega(t_A-t)]s(t-\tau, \varphi, \Phi)dt\right\} \tag{7.10}$$

and for incoherent processing of a small number of repetitions [formulas (6.11) and (6.15)]

$$\begin{aligned} Y(\tau, \theta, \Phi)&=\sum_k \ln I_0(Y_k)\approx \\ &\approx\sum_k L\left\{\frac{2}{N_0}\int_{-\infty}^{\infty} x(t)\gamma[\omega(t_A-t)]s_{T_M}(t-\tau-kT_M, \varphi, \Phi)\, dt\right\}. \end{aligned} \tag{7.11}$$

We shall assume below that a fixed signal with parameters τ_0, $\theta_0(\omega t_{A0})$, Φ_0, ε_0, ϕ_0 is input to the receiver. Then the outputs (7.10) and (7.11) can be represented in a form analogous to (5.26)

$$Y(\tau,\ \theta,\ \Phi)=\begin{cases} 2q^2\varepsilon_0\,|\Psi(\tau-\tau_0,\ \theta-\theta_0,\ \Phi-\Phi_0)|+N_{cx}(\tau,\ \theta,\ \Phi) \\ \text{— within high correlation region,} \\ L[n_1(\tau,\ \theta,\ \Phi,\ \varphi)] \\ \text{— outside high correlation region,} \end{cases} \tag{7.12}$$

where N_{cx} (τ,θ,Φ) and $n_1(\tau,\theta,\Phi,\phi)$ are random processes for which

$$\begin{gathered} \langle N_{cx}(\tau_1,\ \theta_1,\ \Phi_1)\,N_{cx}(\tau_2,\ \theta_2,\ \Phi_2)\rangle= \\ =2q^2\,|\Psi(\tau_1-\tau_2,\ \theta_1-\theta_2,\ \Phi_1-\Phi_2)|; \end{gathered} \tag{7.13}$$

$$\begin{gathered} \langle n_1(\tau_1,\ \theta_1,\ \Phi_1,\ \varphi_1)\,n_1(\tau_2,\ \theta_2,\ \Phi_2,\ \varphi_2)\rangle= \\ =|\Psi(\tau_1-\tau_2,\ \theta_1-\theta_2,\ \Phi_1-\Phi_2)|\cos[2\pi f_0(\tau_1-\tau_2)+ \\ +\chi(\tau_1-\tau_2,\ \Phi_1-\Phi_2,\ \theta_1-\theta_2)+\varphi_1-\varphi_2], \end{gathered} \tag{7.14}$$

and $|\Psi(\tau,\theta,\Phi)|$ and $\chi(\tau,\theta,\Phi)$ are the absolute value and argument of the joint (for the parameters τ,θ,Φ) complex correlation modulation function $\Psi(\tau,\theta,\Phi)$. We shall provide the analytical expressions of the joint modulation correlation function $|\Psi(\tau,\theta,\Phi)|$ entering into (7.12) for the most interesting practical cases.

In the case of a coherent system for processing the received oscillations

$$|\Psi(\tau,\ \theta,\ \Phi)|=\frac{1}{2Q^2}\left|\int_{-\infty}^{\infty}\gamma(\omega t)\,\gamma(\omega t-\theta)\,S(t)\,\overline{S(t-\tau)}\,e^{j2\pi\Phi t}\,dt\right|. \tag{7.15}$$

The signal S(t) in (7.15), as a rule, can be represented as the sum of harmonic components. For example, when Doppler signals are used

$$S(t) = S_0 \cos 2\pi Ft = \frac{1}{2} S_0 e^{j2\pi Ft} + \frac{1}{2} S_0 e^{-j2\pi Ft} \tag{7.16}$$

and direct substitution of (7.16) in (7.15) yields

$$\begin{aligned} |\Psi(\tau, 0, \Phi)| = |\cos 2\pi F\tau|\, |\Psi_\gamma(0, \Phi)| + \\ + \frac{1}{2} |\Psi_\gamma(0, \Phi - 2F)| + \frac{1}{2} |\Psi_\gamma(0, \Phi + 2F)|. \end{aligned} \tag{7.17}$$

Here and below $\Psi_\gamma(\theta,\Phi)$ is determined by the formula

$$\Psi_\gamma(\theta, \Phi) = \frac{\int_{-\infty}^{\infty} \gamma(\omega t)\, \gamma(\omega t - \theta)\, e^{j2\pi\Phi t} dt}{\int_{-\infty}^{\infty} \gamma^2(\omega t)\, dt} \tag{7.18}$$

and is the joint correlation function with respect to the parameters θ and Φ of the directivity pattern or cutoff function $\gamma(\omega t)$.

For a bell-shaped approximation of the directivity pattern

$$\gamma(\theta) = k_0 \exp\left(-2 \ln 2 \frac{\theta^2}{\Delta_\Sigma^2}\right). \tag{7.19}$$

Furthermore

$$|\Psi_\gamma(\theta, \Phi)| = \exp\left(-\ln 2\frac{\theta^2}{\Delta_\Sigma^2}\right)\exp\left(-\frac{\pi^2\Delta_\Sigma^2\Phi^2}{4\omega^2\ln 2}\right). \tag{7.20}$$

In the general case, when

$$S(t) = \sum_{k=-\infty}^{\infty} S_{T_M}(t - kT_M), \tag{7.21}$$

employing the notation from § 5.9

$$\Psi_{T_M}(\tau, \Phi) = \frac{\int_{-\infty}^{\infty} S_{T_M}(t)\overline{S_{T_M}(t-\tau)}\, e^{j2\pi\Phi t}\, dt}{\int_{-\infty}^{\infty} |S_{T_M}(t)|^2\, dt} \tag{7.22}$$

for the joint correlation function of a signal taken over a single repetition period, as well as the notation

$$\Gamma(f) \leftrightarrow \gamma(\omega t) \tag{7.23}$$

and assuming that the adjacent spectralized $\Gamma(f)$ and $\Gamma(f+\frac{1}{T_M})$ practically do not overlap, we find

$$|\Psi(\tau, \theta, \Phi)| = \sum_l \sum_k \left|\Psi_\gamma\left(0, \Phi - \frac{l}{T_M}\right)\Psi_{T_M}\left(\tau - kT_M, \frac{l}{T_M}\right)\right|. \tag{7.24}$$

In the working region $|\tau| < T_M$ and

$$|\Psi(\tau, \theta, \Phi)| = \sum_{l} \left| \Psi_{\gamma}\left(\theta, \Phi - \frac{l}{T_M}\right) \Psi_{T_M}\left(\tau, \frac{l}{T_M}\right) \right|. \tag{7.25}$$

For an incoherent system with a small number of repetitions the correlation function entering in (7.12) is expressed by the formula

$$\begin{aligned} |\Psi(\tau, \theta, \Phi)| = \frac{1}{2Q^2} \sum_{k} \Big| \int_{-\infty}^{\infty} & \gamma(\omega t)\, \gamma(\omega t - \theta)\, S_{T_M}(t - kT_M) \times \\ & \times \overline{S_{T_M}(t - kT_M - \tau)}\, e^{j2\pi\Phi t}\, dt \Big|. \end{aligned} \tag{7.26}$$

Using the same notation and assumptions as in the preceding case, (7.26) becomes

$$|\Psi(\tau, \theta, \Phi)| = |\Psi_{\gamma}(\theta, 0)|\, |\Psi_{T_M}(\tau, \Phi)|. \tag{7.27}$$

Thus, in systems which search (scan) the range coordinate τ and the angular coordinate Φ jointly, the joint modulation correlation function $|\Psi(\tau,\theta,\Phi)|$, which determines the potential capabilities of the system, represents the product (or the sum of the products) of the two functions Ψ_{TM} and Ψ_{γ}, one of which includes the parameter τ, and the other -- the parameter θ. Furthermore, the function Ψ_{γ} depends only upon the shape of the system antenna pattern, while the function Ψ_{TM} depends upon the form of the signal. Because of this, the potential system indicators in terms of the angular coordinate θ depend only upon the directivity pattern, and not the form of the signal. On the other hand, the potential range coordinate indicators depend upon the form of the signal, but not the directivity pattern.

With incoherent processing the information about target velocity contained in the set of initial phases of the high frequency oscillations with different repetition periods is lost. The only portion of the velocity information retained is that which is included in one repetition period of the reflected signal. This physically obvious fact is reflected in formula (7.25).

The expressions derived for the joint modulation correlation function make it possible to determine the region of high correlation in the working sector of space (τ, θ, Φ), after which the threshold ratio can be calculated by the corresponding formulas presented in the preceding chapters. The equivalent number of orthogonal signals (or number of resolvable signals) can be assumed to be approximately

$$m \simeq \frac{\text{volume of region of variation of useful parameters } \tau, \theta, \Phi}{\text{volume of region of high correlation in working sector of space } (\tau, \theta, \Phi)} \tag{7.28}$$

or

$$m = m_\tau m_\theta m_\Phi, \tag{7.29}$$

where m_τ, m_θ, m_Φ -- equivalent number of orthogonal signals (number of resolvable positions) along the τ, θ, and Φ axes.

The extent of the region of high correlation $\Delta\theta$, which is a measure of azimuth resolution, can be estimated approximately by the correlation angle

$$\Delta\theta \approx \theta_k = \int_{-\theta_m}^{\theta_m} |\Psi(0, \theta, 0)|^2 d\theta. \tag{7.30}$$

For a bell-shaped approximation of the directivity pattern (7.19)

$$\Psi(0, \theta, 0) = \Psi_\gamma(\theta) = \exp\left(-\ln 2 \frac{\theta^2}{\Delta_\Sigma^2}\right) \tag{7.31}$$

and

$$\theta_k = \sqrt{\frac{\pi}{2 \ln 2}} \Delta_\Sigma \approx 1{,}5 \Delta_\Sigma . \tag{7.32}$$

The problem of determining the potential accuracy of direction finding by flat rotation of the pattern for pulsed incoherent systems is solved in [31] with a number of additional limitations, specifically, assuming that the range coordinate is known. The determination of the optimum output in the region of high correlation which is provided makes it possible to estimate the potential angular coordinate sampling accuracy in more general form.

In accordance with the analytical method employed we shall assume that the oscillation input to the receiver contains a signal with fixed parameters τ_0, θ_0, Φ_0, ε_0, ϕ_0 and that the angular coordinate and range coordinate are estimated jointly. We shall assume that the signal in each repetition period does not have velocity resolution

$$\Psi_{T_M}(\tau, \Phi) = \Psi_{T_M}(\tau, 0) = \Psi_{T_M}(\tau). \tag{7.33}$$

We shall also assume that for coherent systems the sampling is done in a channel for which in (7.25)

$$\Phi - \Phi_0 - \frac{l}{T_M} \approx 0 \tag{7.34}$$

or in (7.17)

$$\Phi - \Phi_0 \approx 0. \tag{7.35}$$

Then the output of an incoherent system with a small number of repetitions, as well as of a coherent system for a frequency channel through which the signal passes, will have the same appearance (4.52) in the region of high correlation

$$Y(\tau, \theta) = 2q^2\varepsilon_0 |\Psi_f(\theta - \theta_0) \Psi_{T_M}(\tau - \tau_0)| + N_c(\tau, \theta), \tag{7.36}$$

where the random process $N_c(\tau,\theta)$ is such that

$$\langle N_c(\tau_1, \theta_1) N_c(\tau_2, \theta_2)\rangle = 2q^2 |\Psi_f(\theta_1 - \theta_2) \Psi_{T_M}(\tau_1 - \tau_2)|. \tag{7.37}$$

The matrix of second moments of the best estimate θ_0^*, τ_0^* is defined by formulas (4.57)-(4.60)

$$\begin{vmatrix} \delta_\tau^2 & \delta_{\tau\theta}^2 \\ \delta_{\tau\theta}^2 & \delta_\theta^2 \end{vmatrix} = \begin{vmatrix} \frac{1}{2q^2\varepsilon_0^2\beta_\tau^2} & 0 \\ 0 & \frac{1}{2q^2\varepsilon_0^2\beta_\theta^2} \end{vmatrix}, \tag{7.38}$$

where

$$\beta_\tau^2 = \left| \frac{\partial^2}{\partial\tau^2} \Psi_{TM}(\tau) \right|_{\tau=0}, \quad \beta_\theta^2 = \left| \frac{\partial^2}{\partial\theta^2} \Psi_T(\theta) \right|_{\theta=0} \tag{7.39}$$

Thus, when the angular coordinate and range coordinate are sampled jointly, the potential dispersions of the estimates or the dispersions of the estimates of the maximum likelihood δ_τ^2 are δ_θ^2 the same as when one of these parameters is estimated with the second parameter known exactly. Accordingly, when the range coordinate is known exactly (and when the parameters are estimated jointly (τ and θ)) the potential sampling accuracy of the angular coordinate for coherent systems and for incoherent systems with a large number of repetitions is

$$\delta_\theta^2 = \frac{1}{2q^2\varepsilon_0^2\beta_\theta^2}. \tag{7.40}$$

When the generalized directivity pattern of the receiving and transmitting antennas are approximated in the form of bell-shaped function (7.19), we find

$$\beta_\theta^2 = \frac{2\ln 2}{\Delta_\Sigma^2}. \tag{7.41}$$

The formulas presented in the present section can also be used to obtain more complex formulas which determine the dispersion of the joint estimate of the parameters τ,θ and Φ.

7.2. Optimal Processing During Azimuth Scanning

We have not yet touched upon the question of optimum processing of the received oscillations during spatial scanning. Our investigation of potential capabilities has been based on the assumption that an optimum output $Y(\tau,\theta,\Phi)$ or $Y(\tau,\theta)$ is formed if the system does not have velocity resolution.

We shall now discuss the simplest, and near-optimum, method for forming the output. This method is based on the fact that two parameters -- range and azimuth -- are determined by the delay time of the modulating and cutoff functions, respectively. A delay search can be made by means of optimum filtering or some equivalent method of carrying out an optimum linear operation.

As a simple example we shall examine a system designed to receive signals reflected from stationary targets. The optimum output of such a system is determined by expression (7.10), assuming that $\Phi = 0$

$$Y(\tau,\ \theta)=L\left[\int\limits_{-\infty}^{\infty} x(t)\gamma(\omega t-\theta)\sum_{k} s_{T_M}(t-\tau-kT_M)\,dt\right]. \qquad (7.42)$$

We shall assume that the receiver consists of a linear system with impulse response

$$h(t)=\gamma(\omega t)\sum_{k} \bar{s}_{T_M}(-t-kT_M) \qquad (7.43)$$

and an envelope detector. In order to make it easier to write the formulas, the time offset in (7.43) is omitted, which makes it possible to satisfy the conditions for implementation of the system. The voltage at the output of such a receiver will appear as

$$Z(t)=L\left\{\int\limits_{-\infty}^{\infty} x(\xi)\gamma[\omega(t-\xi)]\sum_{k} s_{T_M}(\xi-t-kT_M)\,d\xi\right\} \qquad (7.44)$$

and, if we introduce the notation

$$t = nT_M + \tau \quad (0 < \tau < \tau_m;\ n = 0,\ 1,\ 2\ \ldots), \tag{7.45}$$

and

$$Z(nT_M + \tau) = L\left\{\int_{-\infty}^{\infty} x(\xi)\,\gamma\left[\omega(nT_M - \xi) + \omega\tau\right]\sum_k s_{TM}(\xi - \tau - kT_M)\,d\xi\right\}. \tag{7.46}$$

Comparison of (7.46) and (7.42) shows that $Z(nT_M+\tau)$ is the optimum output for discrete values of the angular coordinate (equal to $\omega nT_M+\omega\tau$)

$$Z(t) = Z(nT_M + \tau) = Y(\tau,\ \omega nT_M + \omega\tau). \tag{7.47}$$

The angular offset $\omega\tau$, when significant, can be considered exactly, since it is a known quantity which is determined during independent sampling of the range coordinate.

An analogous result is obtained in systems designed for receiving signals reflected from moving targets.

The optimum filtering method, or other equivalent methods for accomplishing an optimum linear operation, thus make it possible to obtain in a single-channel system an optimum output which is continuous with respect to the parameter τ and discrete with respect to the parameter θ with a digitization interval of $\omega\, T_M$.

In accordance with the general principles for practical realization of the potential capabilities of a system in terms of detection it is necessary that the extent of the region of high correlation be greater than the digitization interval. In order to realize potential capabilities in terms of accuracy in estimating the angular coordinate it is necessary to have at least two values of the output in each interval, equal to the length of the region of high correlation. In other words, in order to realize the potential capabilities of the system when a single-channel arrangement is used to form the output $Z(t)$, the duration of the reflected signal must be at least twice the repetition period

$$T > 2T_{\text{м}}. \qquad (7.48)$$

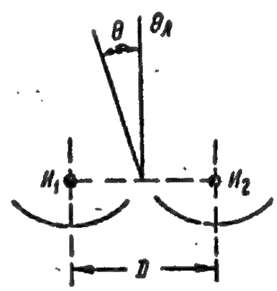
If condition (7.48) is not satisfied, the potential capabilities can be realized in a multi-channel arrangement for forming the output Z(t) with the appropriate time offset between the channels.

7.3. Phase Direction Finding Method

The phase and amplitude direction finding methods are used extensively in radar to estimate the angular coordinate of a detected target. These methods can also be used to form the error signal and, in principle, for azimuth scanning as well. From the viewpoint of antenna engineering, phase and amplitude direction finding systems can be designed in many different ways. We shall not be dealing with the design singularities of antenna systems or other details. In our explanation of the methods, we shall present the most obvious ways of implementing them.

We shall begin by examining the potential angular coordinate sampling accuracy in the phase direction finding method. The antenna system consists of two reflectors, with feeds $И_1$ and $И_2$ at the foci (Fig. 7.1) and separated by distance D. The patterns of both antennas are the same, and have maximum gain in the direction of the geometric axis θ_A.

Fig. 7.1. Antenna arrangement for phase direction finding



In order to simplify the formulas below, it is assumed that the angular position of the geometric axis θ_A coincides with the direction corresponding to the origin of the angular coordinate θ, i.e., $\theta_A = 0$. The angular coordinate is sampled within a relatively small sector $|\theta| \leq \theta_m$, for which the antenna gain can be assumed constant.

The form of the signal з(t) or the form of the modulating function S(t) is of no importance. It is assumed here and below that the reflected signal has fixed intensity and contains only a single repetition period

with duration T. The influence of the fact that the reflected signal contains a greater number of repetition periods and has random intensity can be allowed for easily. This issue was examined in sufficient detail in our study of coherent and incoherent reception methods.

The signals reflected from the target and received by the first and second antennas can be written in the form

$$\begin{aligned} s_1(t-\tau, \theta, \varphi) &= \operatorname{Re}\left\{S(t-\tau)\exp\left[j2\pi f_0(t-\tau)+j\frac{\pi D}{\lambda}\theta+j\varphi\right]\right\}, \\ s_2(t-\tau, \theta, \varphi) &= \operatorname{Re}\left\{S(t-\tau)\exp\left[j2\pi f_0(t-\tau)-j\frac{\pi D}{\lambda}\theta+j\varphi\right]\right\}. \end{aligned} \tag{7.49}$$

In (7.49) sin θ is replaced with θ in accordance with the assumption of the smallness of angle θ, and the initial phase ϕ can contain a term which determines the relationship between the received oscillation and the target velocity.

The investigation is done for internal interference in the receiving channel*. Therefore, interference in the channels of the first and second antennas are mutually independent of random functions. Considering that interference is added mainly during the amplification and frequency conversion of the received signals, we can imagine two processing circuits or methods. In the first method the received signals s_1 and s_2 are amplified directly, converted, etc. In the second system the sum and difference signals

$$\begin{aligned} s_{\Sigma 1}(t-\tau, \theta, \varphi) &= s_1(t-\tau, \theta, \varphi)+s_2(t-\tau, \theta, \varphi)= \\ &= 2\cos\left(\frac{\pi D}{\lambda}\theta\right)\operatorname{Re}\{S(t-\tau)\exp[j2\pi f_0(t-\tau)+j\varphi]\}, \\ s_{\Sigma 2}(t-\tau, \theta, \varphi) &= s_1(t-\tau, \theta, \varphi)-s_2(t-\tau, \theta, \varphi)= \\ &= 2\sin\left(\frac{\pi D}{\lambda}\theta\right)\operatorname{Im}\{S(t-\tau)\exp[j2\pi f_0(t-\tau)+j\varphi]\}, \end{aligned} \tag{7.50}$$

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* This limitation has not been introduced previously.

are first formed in the antenna-feed system, after which the signals $з_{\Sigma 1}$ and $з_{\Sigma 2}$ are amplified, converted, etc.

In accordance with the definition of the parameter estimation problem, we shall assume that the first and second channels receive as input the oscillations $x_1(t)$ and $x_2(t)$, consisting of statistically independent interference $n_1(t)$ and $n_2(t)$ and the signals $з_1$ and $з_2$ (or $з_{\Sigma 1}$ and $з_{\Sigma 2}$) with unknown useful parameters τ and θ. We shall assume that the a priori distribution function of the useful parameters is uniform over the required working region of variation of these parameters, and is equal to zero outside the working region.

The likelihood coefficient $\Lambda(\tau,\theta)$, which differs by a constant coefficient from the function of a posteriori probability of occurrence in the received oscillations x_1 and x_2 of signals with the parameters (τ,θ) is, for the first method,

$$\Lambda(\tau,\theta)=\exp(-2q^2)I_0\left\{\frac{2}{N_0}L\left[\int_{-\infty}^{\infty}x_1(t)з_1(t-\tau,\theta,\varphi)\,dt+\int_{-\infty}^{\infty}x_2(t)з_2(t-\tau,\theta,\varphi)\,dt\right]\right\}, \tag{7.51}$$

where

$$q^2=\frac{1}{N_0}\int_{-\infty}^{\infty}з_i^2(t,\theta,\varphi)\,dt=\frac{1}{2N_0}\int_{-\infty}^{\infty}|S(t)|^2\,dt\quad(i=1,2). \tag{7.52}$$

In writing (7.51) it is assumed that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left[\sum_{i=0}^{n}\int_{-\infty}^{\infty}x_i(t)з_i(t-\tau,\varphi)\,dt\right]d\varphi=$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left\{\cos[\varkappa(\tau)-\varphi]\,L\left[\sum_{i=1}^{n}\int_{-\infty}^{\infty}x_i(t)з_i(t-\tau,0)\,dt\right]\right\}d\varphi.$$

where $\chi(\tau)$ is the argument of the complex quantity

$$\sum_{l=1}^{n} \int_{-\infty}^{\infty} X_l(t) S_l(t-\tau)\, dt.$$

We can use as the optimum output $Y(\tau,\theta)$ the function

$$Y(\tau, \theta) = \frac{2}{N_0} L \left[\int_{-\infty}^{\infty} x_1(t) s_1(t-\tau, \theta, \varphi)\, dt + \right. \\ \left. + \int_{-\infty}^{\infty} x_2(t) s_2(t-\tau, \theta, \varphi)\, dt \right], \tag{7.53}$$

which, assuming that we are dealing with an ensemble of cases in which a signal with parameters (τ_0,θ_0) is input, transformed into

$$Y(\tau, \theta) = 4q^2_{\vartheta_0} |\Psi(\tau-\tau_0)| \left| \cos \frac{\pi D}{\lambda} (\theta - \theta_0) \right| + \\ + \sqrt{2} q \vartheta_{c1}(\tau, \theta) + \sqrt{2} q \vartheta_{c2}(\tau, \theta), \tag{7.54}$$

where

$$\langle [\vartheta_{c1}(\tau_1, \theta_1) + \vartheta_{c2}(\tau_1, \theta_1)] [\vartheta_{c1}(\tau_2, \theta_2) + \vartheta_{c2}(\tau_2, \theta_2)] \rangle = \\ = 2\Psi(\tau_1 - \tau_2) \cos \frac{\pi D}{\lambda} (\theta_1 - \theta_2). \tag{7.55}$$

The joint modulation correlation function for the parameters τ and θ, like in the other cases of joint sampling of the range coordinate and angular coordinate, is the product of the corresponding correlation function

$$\Psi(\tau, \theta) = \Psi(\tau) \Psi_{\gamma}(\theta). \tag{7.56}$$

Furthermore, as was shown above, in order to calculate the potential sampling accuracy of the angular coordinate θ when τ and θ are measured jointly, it can be assumed that the range coordinate τ is known exactly.

Considering that the correlation function of the signal for the angular coordinate

$$|\Psi'_{\gamma}(0)| = \left|\cos\frac{\pi D}{\lambda}\theta\right| \tag{7.57}$$

has a repetition period of λ/D, elimination of ambiguity in the sample requires that the working sector be selected on the basis of the condition

$$2\theta_m \leqslant \frac{\lambda}{D}. \tag{7.58}$$

The optimum output $Y(\tau,\theta)$ has the form of (4.52). Accordingly, we use formula (4.57) or (4.62) to find the potential dispersion of the angular coordinate samples

$$\delta_\theta^2 = \langle(\theta_0^* - \theta_0)^2\rangle = \frac{1}{4q^2\left(\frac{\pi D}{\lambda}\right)^2}. \tag{7.59}$$

Using the second processing method, in which the sum and difference signals are formed in the antenna feed system, the likelihood coefficient $\Lambda(\tau,\theta)$ is

$$\Lambda(\tau, \theta) = \exp(-4q^2) I_0 \left\{ \frac{4}{N_0} L \left[\cos\frac{\pi D}{\lambda}\theta \int_{-\infty}^{\infty} x_1(t) \times \right.\right.$$

$$\times \operatorname{Re}\{S(t-\tau) e^{j2\pi f_0 (t-\tau)}\} dt + \sin\frac{\pi D}{\lambda}\theta \int_{-\infty}^{\infty} x_2(t) \times$$

$$\left.\left. \times \operatorname{Im}\{S(t-\tau) e^{j2\pi f_0 (t-\tau)}\} dt \right]\right\}. \tag{7.60}$$

If we examine an ensemble of cases with fixed parameters (τ_0, θ_0) of the received signal, the expression for the corresponding optimum output $Y(\tau,\theta)$ becomes

$$Y(\tau, \theta) = 8q^2 |\Psi(\tau-\tau_0)| \left| \cos\frac{\pi D}{\lambda}(\theta-\theta_0) \right| + \\ + 2q[\theta_{c1}(\tau, \theta) + \theta_{c2}(\tau, \theta)]. \tag{7.61}$$

The random process $\Theta_{c1}+\Theta_{c2}$ is defined by (7.55). Output (7.61) is the same as (7.54), to within the coefficient 2 of the quantity q^2.

Therefore, the dispersion of the estimates of the angular coordinates will be

$$\delta_\theta^2 = \frac{1}{8q^2\left(\frac{\pi D}{\lambda}\right)^2}. \tag{7.62}$$

Thus, if we can only disregard noise in the antenna feed system, in contrast to noise in the other components of the receiving circuit, the dispersion of the estimates of the coordinate θ is half as large in the second circuit.

7.4. Amplitude Direction Finding Method

We shall examine a simple circuit which implements the method. We shall assume that the antenna system consists of one reflector and two feeds which are offset symmetrically with respect to the focus in the focal plane such that the directions of the maxima of the directivity patterns $\gamma_1(\theta)$ and $\gamma_2(\theta)$ of the first and second feeds are offset accordingly by an angle of $-\alpha$ and $+\alpha$ with respect to the geometric axis of the antenna system, as is shown in Fig. 7.2 and expressed by the replationships

$$\gamma_1(\theta)=\gamma(\theta+\alpha), \quad \gamma_2(\theta)=\gamma(\theta-\alpha). \tag{7.63}$$

It is assumed that the directivity pattern $\gamma(\theta)$ is an even function with its maximum at $\theta=0$.

Within the working sector, which approximately surrounds the region $|\theta| \leq \alpha$, the signals з_1 and з_2 received by the first and second radiators have the same phase and differ only in amplitude

$$\begin{aligned} \text{з}_1(t-\tau, \theta, \varphi)&=\gamma(\theta+\alpha)\,\text{з}(t-\tau, \varphi), \\ \text{з}_2(t-\tau, \theta, \varphi)&=\gamma(\theta-\alpha)\,\text{з}(t-\tau, \varphi). \end{aligned} \tag{7.64}$$

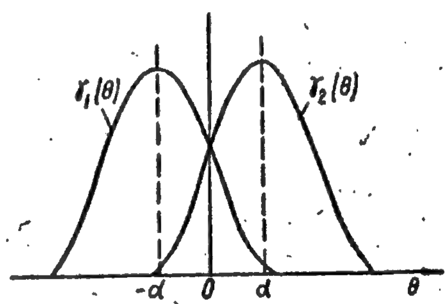

Fig. 7.2. Directivity pattern of radiators $И_1$ and $И_2$ for amplitude direction finding method

The likelihood coefficient for the parameters (τ,θ) using the first processing method (without forming the sum and difference signals) is

$$\Lambda(\tau,\theta)=\exp\left\{-\frac{q^2}{\gamma^2(\alpha)}[\gamma^2(\theta+\alpha)+\gamma^2(\theta-\alpha)]\right\}\times$$
$$\times I_0\left\{\frac{2}{N_0}L\left[\gamma(\theta+\alpha)\int_{-\infty}^{\infty}x_1(t)s(t-\tau,\varphi)\,dt+\right.\right.$$
$$\left.\left.+\gamma(\theta-\alpha)\int_{-\infty}^{\infty}x_2(t)s(t-\tau,\varphi)\,dt\right]\right\}, \tag{7.65}$$

where q^2 denotes

$$q^2=\frac{\gamma^2(\alpha)}{N_0}\int_{-\infty}^{\infty}s^2(t)\,dt. \tag{7.66}$$

We then compose the likelihood equation, assuming that a signal is input with parameters (τ_0, θ_0) and that the range coordinate τ_0 is known in advance

$$\left|\frac{\partial}{\partial\theta}\ln\Lambda(\tau,\theta)\right|_{\tau=\tau_0}\approx\frac{2q^2}{\gamma^2(\alpha)}[-\gamma'(\theta_0^*+\alpha)\gamma(\theta_0^*+\alpha)-$$
$$-\gamma'(\theta_0^*-\alpha)\gamma(\theta_0^*-\alpha)+\gamma'(\theta_0^*+\alpha)\gamma(\theta_0+\alpha)+\gamma'(\theta_0^*-\alpha)\gamma(\theta_0-\alpha)]+$$
$$+\frac{\sqrt{2}q}{\gamma(\alpha)}[\gamma'(\theta_0^*+\alpha)\theta_{c1}+\gamma'(\theta_0^*-\alpha)\theta_{c2}]=0.$$

We expand the functions $\gamma(\tau_0+\alpha)$ and $\gamma(\tau_0-\alpha)$ in the vicinity of the estimate by powers of the small parameter $(\theta_0^*-\theta_0)$. Then the likelihood equation becomes

$$\frac{2q^2}{\gamma^2(\alpha)}(\theta_0^*-\theta_0)\{[\gamma'(\theta_0^*+\alpha)]^2+[\gamma'(\theta_0^*-\alpha)]^2\}+$$
$$+\theta\frac{\sqrt{2}q}{\gamma(\alpha)}\sqrt{[\gamma'(\theta_0^*+\alpha)]^2+[\gamma'(\theta_0^*-\alpha)]^2}=0, \tag{7.67}$$

where Θ is a normal normalized random quantity. Assuming that

$$[\gamma'(\theta_0^*+\alpha)]^2+[\gamma'(\theta_0^*-\alpha)]^2\approx 2[\gamma'(\alpha)]^2, \tag{7.68}$$

we obtain this expression for the error dispersion:

$$\delta_\theta^{*} = \langle(\theta_0^{*} - \theta_0)^2\rangle = \frac{1}{4q^2}\left[\frac{\gamma(\alpha)}{\gamma'(\alpha)}\right]^2. \tag{7.69}$$

With the second processing method, in which the antenna-feed system forms the sum and difference signals,

$$\begin{aligned} s_{\Sigma 1}(t-\tau,\ \theta,\ \varphi) &= [\gamma(\theta+\alpha)+\gamma(\theta-\alpha)]\, s(t-\tau,\ \varphi), \\ s_{\Sigma 2}(t-\tau,\ \theta,\ \varphi) &= [\gamma(\theta+\alpha)-\gamma(\theta-\alpha)]\, s(t-\tau,\ \varphi). \end{aligned} \tag{7.70}$$

The likelihood coefficient is

$$\begin{aligned} \Lambda(\tau,\ \theta) = \exp\left\{-\frac{2q^2}{\gamma^2(\alpha)}[\gamma^2(\theta+\alpha)+\gamma^2(\theta-\alpha)]\right\}\times \\ \times I_0\left\{\frac{2}{N_0} L\left[\{\gamma(\theta+\alpha)+\gamma(\theta-\alpha)\}\int_{-\infty}^{\infty} x_1(t)\, s(t-\tau,\ \varphi)\,dt + \right.\right. \\ \left.\left. + \{\gamma(\theta+\alpha)-\gamma(\theta-\alpha)\}\int_{-\infty}^{\infty} x_2(t)\, s(t-\tau,\ \varphi)\,dt\right]\right\}. \end{aligned} \tag{7.71}$$

If we assume that the received oscillations x_1 and x_2 contain a signal with parameters τ_0 and θ_0, the argument of the function I_0 in (7.71) can be transformed as follows:

$$\begin{aligned} \frac{2q^2}{\gamma^2(\alpha)}\Big\{[\gamma(\theta+\alpha)+\gamma(\theta-\alpha)][\gamma(\theta_0+\alpha)+\gamma(\theta_0-\alpha)]\,|\Psi(\tau-\tau_0)| + \\ + [\gamma(\theta+\alpha)-\gamma(\theta-\alpha)][\gamma(\theta_0+\alpha)-\gamma(\theta_0-\alpha)]\,|\Psi(\tau-\tau_0)| + \\ + \frac{\sqrt{2}\,q}{\gamma^2(\alpha)}\{[\gamma(\theta+\alpha)+\gamma(\theta-\alpha)]\,\theta_1(\tau) + [\gamma(\theta+\alpha)-\gamma(\theta-\alpha)]\,\theta_2(\tau)\} = \\ = \frac{4q^2}{\gamma^2(\alpha)}\,|\Psi(\tau-\tau_0)|\,[\gamma(\theta+\alpha)\gamma(\theta_0+\alpha)+\gamma(\theta-\alpha)\gamma(\theta_0-\alpha)] + \\ + \frac{2q}{\gamma(\alpha)}[\gamma(\theta+\alpha)\,\theta_{c1}(\tau)+\gamma(\theta-\alpha)\,\theta_{c2}(\tau)], \end{aligned}$$

which again, to within the coefficient $\sqrt{2}$ of the first term, leads to a likelihood equation of the type (7.67). Accordingly, for processing in which the sum and difference signals are formed

$$\delta_\theta^2 = \frac{1}{8q^2}\left[\frac{\gamma(\alpha)}{\gamma'(\alpha)}\right]^2. \tag{7.72}$$

Like with the phase direction finding method, if the noise in the antenna feed system can be disregarded as compared with the noise in the other elements of the receiving channel, the use of the second arrangement in which the sum and difference signals are formed makes it possible to reduce the dispersion of the estimates by a factor of 2.

In order to illustrate the formulas derived above, we shall calculate the potential angular coordinate estimation accuracy for the first processing method for an antenna system with a bell-shaped directivity pattern (7.19) and with a pattern width (at the half-power level) of Δ. Here

$$\delta_\theta^2 = \frac{\Delta^4}{4q^2(4\alpha \ln 2)^2}, \tag{7.73}$$

and assuming that the patterns intersect at the half-power level ($\alpha = \frac{\Delta}{2}$), we find

$$\delta_\theta^2 = \frac{\Delta^2}{4q^2(2\ln 2)^2}. \tag{7.74}$$

It is interesting to compare the accuracy with which the angular coordinates are sampled in the phase and amplitude direction finding methods. We shall assume in both cases that the field intensity of the reflected signal, as well as the intensity of the interference in the receiving channels, are the same and, as Fig. 7.3 shows, the total aperture area of the antenna system employed in the phase method (Φ) is equal to the aperture area of the antenna employed in the amplitude method (A). Then the total power of the signals received by radiators $И_1$ and $И_2$ are the same in both systems. Accordingly, the power ratios q^2 which figure in the expressions for potential sampling

accuracy are the same for both direction finding methods. Employing the approximate relationship [29]

$$\Delta \approx 1{,}22 \frac{\lambda}{D_\Sigma} [\text{рад}], \tag{7.75}$$

we can write instead of (7.74)

$$\delta_\theta^2 = \frac{1}{4q^2 \left(1{,}14 \frac{D_\Sigma}{\lambda}\right)^2}, \tag{7.76}$$

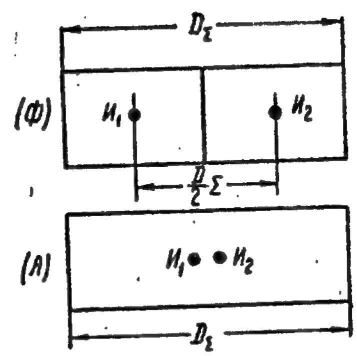

Fig. 7.3. Comparison of potential capabilities of direction finding methods. Aperture areas of antenna systems for phase (Φ) and amplitude (A) direction finding methods.

On the basis of (7.59) and Fig. 7.3 we have

$$\delta_\theta^2 = \frac{1}{4q^2 \left(1{,}57 \frac{D_\Sigma}{\lambda}\right)^2}. \tag{7.77}$$

In the phase direction finding method.

The comparison above is approximate; even so, it indicates definitely that the potential angular coordinate sampling accuracy for both methods is of the same order and that, for a given reflected signal field intensity, it depends basically on the size of the aperture of the antenna system.

7.5. Optimal Processing Schemes for Amplitude and Phase Direction Finding Methods

It was pointed out in Chapter 4 that in order to obtain the best estimate of a signal parameter there is no need to form an optimum output for each value of the measured parameter. Considering the random nature of signal intensity, in order to sample the parameter θ it is sufficient to form two values of the optimum output, after which the best estimate is obtained by solving a system of equations. In defining the optimum scheme we shall assume that, in accordance with the proofs in § 4.2, the likelihood coefficient

$$\Lambda(\tau,\theta,\varepsilon)=\exp\left\{-\frac{\varepsilon^2}{N_0}\left[\int_{-\infty}^{\infty}s_1^2(t,\theta,\varphi)\,dt+\int_{-\infty}^{\infty}s_2^2(t,\theta,\varphi)\,dt\right]\right\}\times$$
$$\times I_0\left\{\frac{2\varepsilon}{N_0}L\left[\int_{-\infty}^{\infty}x_1(t)\,s_1(t-\tau,\theta,\varphi)\,dt+\int_{-\infty}^{\infty}x_2(t)\,s_2(t-\tau,\theta,\varphi)\,dt\right]\right\} \tag{7.78}$$

in the vicinity of the estimate $(\tau_0^*, \theta_0^*, \varepsilon_0^*)$ can be represented as

$$\Lambda(\tau,\theta,\varepsilon)=\exp\left\{-\frac{\varepsilon^2}{N_0}\left[\int_{-\infty}^{\infty}s_1^2(t,\theta,\varphi)\,dt+\int_{-\infty}^{\infty}s_2^2(t,\theta,\varphi)\,dt\right]\right\}\times$$
$$\times I_0\left\{\frac{2\varepsilon\varepsilon_0^*}{N_0}L\left[\int_{-\infty}^{\infty}s_1(t-\tau,\theta,\varphi)\,s_1(t-\tau_0^*,\theta_0^*,\varphi_0^*)\,dt+\right.\right.$$
$$\left.\left.+\int_{-\infty}^{\infty}s_2(t-\tau,\theta,\varphi)\,s_2(t-\tau_0^*,\theta_0^*,\varphi_0^*)\,dt\right]\right\}. \tag{7.79}$$

When a processing scheme is employed in which sum and difference signals are formed, $\text{з}_{\Sigma 1}$ and $\text{з}_{\Sigma 2}$ must be substituted for з_1 and з_2 in (7.78) and (7.79).

The method for constructing the error signal and estimate formation circuit was described in § 4.4. For direction finding systems this method consists of the following. On the basis of (7.79) operations are selected which convert the function $\Lambda(\tau,\theta,\varepsilon)$ for some fixed values of θ and ε to the output $d(\tau)$ which, for a value of τ equal to the true delay time τ_0, represents a monotonic function of the deviation of the direction of the geometric axis from the best estimate θ_0^*. The error signal $d(\tau_0)$, in particular, can be proportional to the deviation of the geometric axis from the best estimate.

In order to obtain the estimate θ_0^* it is necessary to define a second conversion of the function $\Lambda(\tau,\theta,\varepsilon)$ to an additional output $\mathscr{D}(\tau)$, whose knowledge for $\tau=\tau_0$ would make it possible to eliminate the random intensity parameter from the error signal. We shall designate the conversions of the function $\Lambda(\tau,\theta,\varepsilon)$ which we have selected by the symbols Γ_1 and Γ_2, and the fixed values of the parameters θ and ε in the first and second conversions by θ_1, ε_1 and θ_2, ε_2, respectively. Then solving the system of equations

$$\begin{aligned} \Gamma_1[\Lambda(\tau,\ \theta,\ \varepsilon)]_{\varepsilon=\varepsilon_1,\ \theta=\theta_1,\ \tau=\tau_0} &= d(\tau_0), \\ \Gamma_2[\Lambda(\tau,\ \theta,\ \varepsilon)]_{\varepsilon=\varepsilon_2,\ \theta=\theta_2,\ \tau=\tau_0} &= \mathscr{D}(\tau_0) \end{aligned} \tag{7.80}$$

makes it possible to obtain the estimate θ_0^* in the form of some known function ϕ of the variables $d(\tau_0)$ and $\mathscr{D}(\tau_0)$

$$\theta_0^* = \varphi[d(\tau_0),\ \mathscr{D}(\tau_0)]. \tag{7.81}$$

The receiver must, consequently, normalize the output $d(\tau)$ and $\mathscr{D}(\tau)$. The operations over the received oscillations $x_1(t)$ and $x_2(t)$ and the corresponding radio circuits needed to do this are defined on the basis of representation (7.78). The outputs are then input to a decision device whose functions are determined through (7.81).

It was noted in § 4.4 that there is no single solution to the problem of composing the system of equations for obtaining the best estimate. A large number of systems of equations, and accordingly of radio circuits, can be proposed which provide potential sampling accuracy. The designer's problem is obviously to select the system of equations (7.80) such that the corresponding radio circuit has certain practical advantages, such as simplicity of implementation. The choice of systems of equations (7.80) in the present section is made so that the circuits which are obtained are as close as possible to those actually used in practice.

In the phase direction finding method, employing a processing scheme without forming some indifferent signals, we accordingly use the following as the output of the channel which forms the error signal $d(\tau)$ and additional output for obtaining the estimate $\mathfrak{D}(\tau)$:

$$d(\tau)=\left|\frac{\partial}{\partial\theta}[\ln\Lambda(\tau,\ \theta,\ \varepsilon)]^2\right|_{\varepsilon=1,\ \theta=0},$$
$$\mathfrak{D}(\tau)=\left|\frac{\partial^2}{\partial\theta^2}[\ln\Lambda(\tau,\ \theta,\ \varepsilon)]^2\right|_{\varepsilon=1,\ \theta=0}. \qquad (7.82)$$

We shall demonstrate that the relationships stipulated above between the quantities $d(\tau_0)$ and $\mathfrak{D}(\tau_0)$ and the estimate are obtained in this case. If we consider that the square of the envelope of the sum of two oscillating processes can be represented as

$$\{L[f_1(t)\cos(\omega_0 t+\varphi_1)+f_2(t)\cos(\omega_0 t+\varphi_2)]\}^2=$$
$$=f_1^2(t)+f_2^2(t)+2f_1(t)f_2(t)\cos(\varphi_1-\varphi_2),$$

after substituting (7.79) in (7.82), we find

$$d(\tau_0)=8(\varepsilon_0^*)^2q^4\left(\frac{2\pi D}{\lambda}\right)\sin\frac{2\pi D}{\lambda}\theta_0^*,$$
$$\mathfrak{D}(\tau_0)=8(\varepsilon_0^*)^2q^4\left(\frac{2\pi D}{\lambda}\right)^2\cos\frac{2\pi D}{\lambda}\theta_0^* \qquad (7.83)$$

and the expression for obtaining the estimate

$$\theta_0^*\doteq\frac{\lambda}{2\pi D}\operatorname{arc\,tg}\frac{2\pi D}{\lambda}\frac{d(\tau_0)}{\mathfrak{D}(\tau_0)}. \qquad (7.84)$$

We now define the circuit for forming the outputs (7.82), for which we substitute in these expressions the likelihood coefficient (7.78)

$$d(\tau)=\frac{\partial}{\partial\theta}\left\{L\left[\frac{2}{N_0}\int_{-\infty}^{\infty}x_1(t)\,\mathrm{Re}\{S(t-\tau)e^{j2\pi f_0(t-\tau)+j\frac{\pi D}{\lambda}\theta}\}dt+\right.\right.$$
$$\left.\left.+\frac{2}{N_0}\int_{-\infty}^{\infty}x_2(t)\,\mathrm{Re}\{S(t-\tau)e^{j2\pi f_0(t-\tau)-j\frac{\pi D}{\lambda}\theta}\}dt\right]\right\}^2_{\theta=0}= \tag{7.85}$$
$$=\frac{\partial}{\partial\theta}\left\{2Y_1(\tau)Y_2(\tau)\cos\left[\varkappa_1(\tau)-\varkappa_2(\tau)+\frac{2\pi D}{\lambda}\theta\right]\right\}_{\theta=0},$$

where $Y_1(\tau)$, $Y_2(\tau)$ and $x_1(\tau)$ and $x_2(\tau)$ are the envelopes and phases of the high frequency processes

$$\frac{2}{N_0}\int_{-\infty}^{\infty}x_1(t)\,s(t-\tau)\,dt \qquad \text{and} \qquad \frac{2}{N_0}\int_{-\infty}^{\infty}x_2(t)\,s(t-\tau)\,dt. \tag{7.86}$$

Thus,

$$d(\tau)=2\frac{2\pi D}{\lambda}Y_1(\tau)Y_2(\tau)\sin[\varkappa_1(\tau)-\varkappa_2(\tau)]. \tag{7.87}$$

Analogously,

$$\mathcal{D}(\tau)=2\left(\frac{2\pi D}{\lambda}\right)^2Y_1(\tau)Y_2(\tau)\cos[\varkappa_1(\tau)-\varkappa_2(\tau)] \tag{7.88}$$

and on the basis of (7.84)

$$\theta_0^*=\frac{\lambda}{2\pi D}[\varkappa_1(\tau_0)-\varkappa_2(\tau_0)]. \tag{7.89}$$

The optimum processing scheme consists of optimal filtering of the received oscillations $x_1(t)$ and $x_2(t)$ in linear systems with identical impulse responses

$$h_1(t) = h_2(t) = cs(-t,\ \varphi) \tag{7.90}$$

and measuring the phase difference between the oscillations at the output of the optimal filters. Figure 7.4 shows a processing circuit which is a direct interpretation of these expressions. The notation in the diagram is as follows: FD -- phase detector, FI -- phase inverting network, which multiplies the phase by $\pi/2$, DD -- decision device, which forms the estimate θ_0^* on the basis of (7.84).

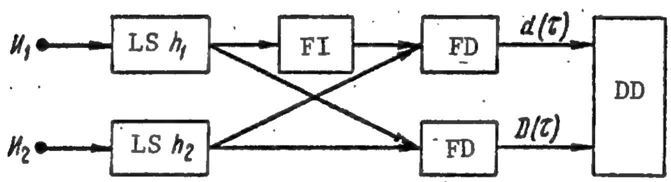

Fig. 7.4. Optimum processing circuit for phase direction finding method

As a check, we shall calculate the dispersion of the estimate (7.89) obtained in the optimum processing circuit. We shall assume that the input is a signal with the parameters $(\tau_0,\ \theta_0,\ \varepsilon_0,\ \phi_0)$. Then the oscillation at the input of the optimal filter will be

$$\frac{2}{N_0}\int\limits_{-\infty}^{\infty} x_1(t)\, s(t-\tau)\, dt \doteq 2q^2 \varepsilon_0 |\Psi(\tau-\tau_0)| \times$$
$$\times \cos\left[2\pi f_0(\tau-\tau_0)+\varphi_0+\frac{\pi D}{\lambda}\theta_0\right]+$$
$$+N_{c1}(\tau)\cos\left[2\pi f_0(\tau-\tau_0)+\varphi_0+\frac{\pi D}{\lambda}\theta_0\right]+ \quad (7.91)$$
$$+N_{s1}(\tau)\sin\left[2\pi f_0(\tau-\tau_0)+\varphi_0+\frac{\pi D}{\lambda}\theta_0\right].$$

Since the signal is significantly stronger than the interference in the vicinity of τ_0, we can write

$$x_1(\tau_0) = \varphi_0 + \frac{\pi D}{\lambda}\theta_0 + \frac{N_{s1}(\tau_0)}{2q^2\varepsilon_0}. \quad (7.92)$$

Analogously,

$$x_2(\tau_0) = \varphi_0 - \frac{\pi D}{\lambda}\theta_0 + \frac{N_{s2}(\tau_0)}{2q^2\varepsilon_0}, \quad (7.93)$$

whence according to (7.89)

$$\theta_0^* = \theta_0 + \frac{N_{s1}(\tau)+N_{s2}(\tau)}{2q^2\varepsilon_0}\frac{\lambda}{2\pi D} \quad (7.94)$$

and

$$\delta_\theta^2 = \langle(\theta_0^* - \theta_0)^2\rangle = \frac{1}{4q^2\varepsilon_0^2\left(\frac{\pi D}{\lambda}\right)^2}. \quad (7.95)$$

As might be expected, the dispersion of the estimates obtained in the optimum processing circuit is the same as the potential dispersion of the estimates (7.59), if we only allow for the random nature of the intensity of the received signals in (7.59). The coincidence of these

dispersions need not be checked below.

In the amplitude direction finding method, employing a processing circuit which does not form sum and difference signals, the output of the channel which forms the error signal $d(\tau)$ and the additional output $\mathfrak{D}(\tau)$ can be

$$d(\tau)=\frac{1}{\gamma'(\alpha)}\left|\frac{\partial}{\partial\theta}\ln\Lambda(\tau,\theta,\varepsilon)\right|_{\varepsilon=1,\,\theta=0} \tag{7.96}$$

and

$$\mathfrak{D}(\tau)=\frac{1}{\gamma''(\alpha)}\left|\frac{\partial^2}{\partial\theta^2}\left\{\ln\Lambda(\tau,\theta,\varepsilon)+\right.\right.$$
$$\left.\left.+\frac{q^2}{\gamma^2(\alpha)}[\gamma^2(\theta+\alpha)+\gamma^2(\theta-\alpha)]\right\}\right|_{\varepsilon=1,\,\theta=0}. \tag{7.97}$$

According to the expression for the likelihood coefficient in the vicinity of the estimate (7.79)

$$d(\tau_0)=\frac{2q^2\varepsilon_0^*}{\gamma^2(\alpha)}[\gamma(\theta_0^*+\alpha)-\gamma(\theta_0^*-\alpha)], \tag{7.98}$$

$$\mathfrak{D}(\tau_0)=\frac{2q^2\varepsilon_0^*}{\gamma^2(\alpha)}[\gamma(\theta_0^*+\alpha)+\gamma(\theta_0^*-\alpha)]. \tag{7.99}$$

The estimate θ_0^* can be found by solving (7.98) and (7.99) simultaneously.

In order to define the optimum processing circuit, we substitute (7.78) in (7.96) and (7.98). Since the signals in both channels are in phase

$$\frac{2}{N_0}L\left[\gamma(\theta+\alpha)\int_{-\infty}^{\infty}x_1(t)s(t-\tau)dt+\right.$$
$$\left.+\gamma(\theta-\alpha)\int_{-\infty}^{\infty}x_2(t)s(t-\tau)dt\right]\approx\gamma(\theta+\alpha)Y_1(\tau)+\gamma(\theta-\alpha)Y_2(\tau), \tag{7.100}$$

where $Y_1(\tau)$ and $Y_2(\tau)$, as above, are the envelopes of processes (7.86). Therefore,

$$d(\tau) = Y_1(\tau) - Y_2(\tau), \tag{7.101}$$

$$\mathcal{D}(\tau) = Y_1(\tau) + Y_2(\tau). \tag{7.102}$$

Figure 7.5 shows the corresponding processing circuit. We note that this circuit does not respond to variations in the initial phase of the oscillations. The initial phases of the oscillations of the received signals s_1 and s_2 can be different, or even statistically independent.

For this reason, it is possible for this circuit to employ a single processing channel, with the first and second radiators connected to it alternately for a single repetition period.

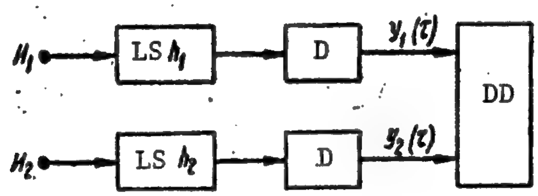

Fig. 7.5. Optimum processing circuit for amplitude direction finding method

In the case of processing in which sum and difference signals are formed, the question of the structure of the optimal receiving circuit will be considered in more general form, for phase and amplitude direction finding simultaneously. We note that for both direction finding methods

$$s_1(t-\tau, \theta, \varphi) = s_2(t-\tau, -\theta, \varphi) \equiv s(t-\tau, 0, \varphi), \tag{7.103}$$

$$\lim_{\theta\to 0} \frac{1}{2\theta} s_{\Sigma 2}(t-\tau, \theta, \varphi) = \left| \frac{\partial}{\partial\theta} s(t-\tau, \theta, \varphi) \right|_{\theta=0} \equiv s'_\theta(t-\tau, 0, \varphi). \tag{7.104}$$

By analogy with the preceding case, the output of the error signal formation channel can be the function

$$d(\tau) = c \left| \frac{\partial}{\partial \theta} \ln \Lambda(\tau, \theta, \varepsilon) \right|_{\varepsilon=1, \theta=0} =$$
$$= c \left| \frac{\partial}{\partial \theta} L \left[\int_{-\infty}^{\infty} x_1(t) s_{\Sigma 1}(t-\tau, \theta, \varphi)\, dt + \right.\right.$$
$$\left.\left. + \int_{-\infty}^{\infty} x_2(t) s_{\Sigma 2}(t-\tau, \theta, \varphi)\, dt \right] \right|_{\theta=0}. \tag{7.105}$$

Further transformation of (7.105) considering (7.104) allows us to obtain

$$c \lim_{\theta \to 0} \frac{1}{2\theta} \left\{ L \left[\int_{-\infty}^{\infty} x_1(t) s_{\Sigma 1}(t-\tau, \theta, \varphi)\, dt + \right.\right.$$
$$\left. + \int_{-\infty}^{\infty} x_2(t) s_{\Sigma 2}(t-\tau, \theta, \varphi)\, dt \right] - L \left[\int_{-\infty}^{\infty} x_1(t) s_{\Sigma 1}(t-\tau, -\theta, \varphi)\, dt + \right.$$
$$\left.\left. + \int_{-\infty}^{\infty} x_2(t) s_{\Sigma 2}(t-\tau, -\theta, \varphi)\, dt \right] \right\} =$$
$$= c \left\{ L \left[\lim_{\theta \to 0} \frac{1}{\theta} \int_{-\infty}^{\infty} x_1(t) s(t-\tau, 0, \varphi)\, dt + \right.\right.$$
$$\left. + \int_{-\infty}^{\infty} x_2(t) s_\theta'(t-\tau, 0, \varphi)\, dt \right] -$$
$$- L \left[\lim_{\theta \to 0} \frac{1}{\theta} \int_{-\infty}^{\infty} x_1(t) s(t-\tau, 0, \varphi)\, dt - \right.$$
$$\left.\left. - \int_{-\infty}^{\infty} x_2(t) s_\theta'(t-\tau, 0, \varphi)\, dt \right] \right\}. \tag{7.106}$$

If

$$L\left[\int_{-\infty}^{\infty} x_1(t)\, s(t-\tau,\ 0,\ \varphi)\, dt\right] \gg L\left[\int_{-\infty}^{\infty} x_2(t)\, s'_\theta(t-\tau,\ 0,\ \varphi)\, dt\right] \qquad (7.107)$$

(which, of course, occurs with both direction finding methods near $\tau=\tau_0$), the occurrence of a large coefficient ahead of the first terms in the square brackets has practically no influence on the result, since these terms compensate one another after the operation L has been executed. We can therefore write

$$\begin{aligned} d(\tau) = L\Bigg[c_1 \int_{-\infty}^{\infty} x_1(t)\, s(t-\tau,\ 0,\ \varphi)\, dt + \\ + c_2 \int_{-\infty}^{\infty} x_2(t)\, s'_\theta(t-\tau,\ 0,\ \varphi)\, dt\Bigg] - \\ - L\Bigg[c_1 \int_{-\infty}^{\infty} x_1(t)\, s(t-\tau,\ 0,\ \varphi)\, dt - \\ - c_2 \int_{-\infty}^{\infty} x_2(t)\, s'_\theta(t-\tau,\ 0,\ \varphi)\, dt\Bigg], \end{aligned} \qquad (7.108)$$

where c_1 and c_2 are arbitrary constants, but are such that adding them as factors to the right and left parts of (7.107), respectively, does not violate the inequality.

We assume the additional output $\mathfrak{D}(\tau)$ to be

$$\begin{aligned} \mathfrak{D}(\tau) = c\Big/ \ln \Lambda(\tau,\ \theta,\ \varepsilon) + \frac{1}{N_0}\int\left(s_{\Sigma 1}^2 + s_{\Sigma 2} t\right) dt \Big/_{\varepsilon=1,\ \theta=0} = \\ = cL\left[\frac{4}{N_0}\int_{-\infty}^{\infty} x_1(t)\, s(t-\tau,\ 0,\ \varphi)\, dt\right], \end{aligned} \qquad (7.109)$$

which can be written as

$$\begin{aligned}\mathcal{D}(\tau) = L\Big[c_1 \int_{-\infty}^{\infty} x_1(t)\, s(t-\tau,\ 0,\ \varphi)\, dt &+ \\ + c_2 \int_{-\infty}^{\infty} x_2(t)\, s_0'(t-\tau,\ 0,\ \varphi)\, dt\Big] &+ \\ + L\Big[c_1 \int_{-\infty}^{\infty} x_1(t)\, s(t-\tau,\ 0,\ \varphi)\, dt &- \\ - c_2 \int_{-\infty}^{\infty} x_2(t)\, s_0'(t-\tau,\ 0,\ \varphi)\, dt\Big]&.\end{aligned} \tag{7.110}$$

Figure 7.6 shows an optimum processing circuit which forms (7.108) and (7.110). The component Σ_1 and Σ_2 shown in the figure form the sum (+) and difference (-) of the input oscillations. The impulse responses $h_1(t)$ and $h_2(t)$ of the linear systems must be, to within a constant coefficient:

-- for the amplitude method

$$h_1(t) = h_2(t) = s(t_0 - t,\ 0,\ \varphi) \tag{7.111}$$

-- for the phase method

$$\begin{aligned} h_1(t) &= \mathrm{Re}\,\{S(t_0 - t)\exp[j(2\pi f_0 t + \varphi)]\}, \\ h_2(t) &= \mathrm{Im}\,\{S(t_0 - t)\exp[j(2\pi f_0 t + \varphi)]\}. \end{aligned} \tag{7.112}$$

The functions $Y_1(\tau)$ and $Y_2(\tau)$ input to the decision device (DD) in Fig. 7.6 are connected with the outputs $d(\tau)$ and $\mathcal{D}(\tau)$ as

$$\begin{aligned} d(\tau) &= Y_1(\tau) - Y_2(\tau), \\ \mathcal{D}(\tau) &= Y_1(\tau) + Y_2(\tau). \end{aligned}$$

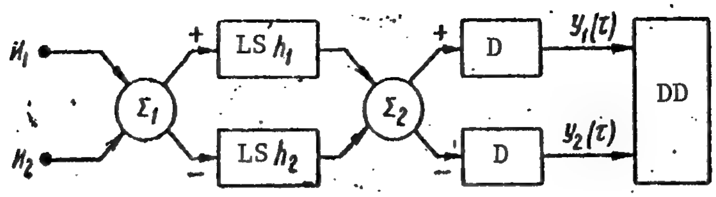

Fig. 7.6. Optimum processing circuit for phase and amplitude direction finding forming sum and difference signals

The equations for obtaining the estimates are analogous to the corresponding equations in circuits which do not form the sum and difference signals. If we again use representation (7.79) for the likelihood coefficient in the vicinity of the estimate, the equations can be transformed as follows:

-- for the phase direction finding method

$$d(\tau_0) = c \sin \frac{\pi D}{\lambda} \vartheta_0^*, \tag{7.113}$$

$$\mathcal{D}(\tau_0) = c \cos \frac{\pi D}{\lambda} \vartheta_0^*, \tag{7.114}$$

-- for the amplitude direction finding method

$$d(\tau_0) = c\,[\gamma(\vartheta_0^* + \alpha) - \gamma(\vartheta_0^* - \alpha)], \tag{7.115}$$

$$\mathcal{D}(\tau_0) = c\,[\gamma(\vartheta_0^* + \alpha) + \gamma(\vartheta_0^* - \alpha)]. \tag{7.116}$$

The circuits obtained in this section, with the possible exception of that in Fig. 7.6, are not new. But even so, it is interesting, first, that these circuits were obtained by applying the analytical apparatus for defining optimal circuits and, second, that it has been proved that the circuits in Figs. 7.4, 7.5 and 7.6 exhaust the potential capabilities of amplitude and phase direction finding systems.

7.6. Multichannel Systems

Amplitude and phase systems are designed primarily for forming an error signal and sampling the angular coordinate. However, they can also be used for detection as well. One advantage is their capability of operating with stationary antenna systems. Even so, they have serious deficiencies: the received signals, which differ only in angular coordinate, are strongly correlated. In other words, amplitude and phase systems have no azimuth resolution. Another deficiency is the extremely small size of the working sector, which is determined for the amplitude method by the width of the directivity pattern, and for the phase method by the sector of indeterminacy, the angular extent of which is λ/D rad. These deficiencies can be reduced in principle to some extent by employing multi-element and multichannel systems [73, 74].

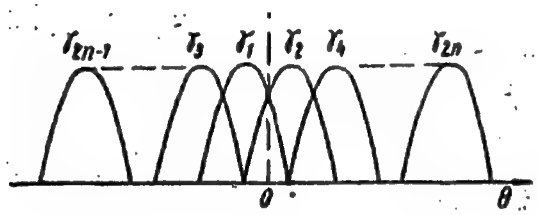

Fig. 7.7. Directivity patterns of radiators $И_1, И_2, \ldots, И_{2n}$ for multichannel amplitude direction finding method

With the amplitude method, a multichannel system consists of some number, such as 2n, of radiators, the relative positioning of whose directivity patterns $\gamma_1(\theta)$, $(\theta), \ldots\ldots, \gamma_2{}^n(\theta)$ is shown in Fig. 7.7. The signals received by the radiators are input to 2n independent processing channels. Each received signal is passed to the output of two channels. The angular coordinate is determined by the numbers of these channels and the ratio of the amplitudes of the signals in them.

In studying the phase method we shall again examine the simple example in which the antenna system is a linear array (Fig. 7.8) consisting of 2n reflectors with radiators $И_1$, $И_2$, $\ldots, И_{2n}$ arranged at their foci. The signal with parameters (τ, θ, ϕ) received by radiators $И_{2i-1}$ and $И$ can be written as

$$з_{2i-1}(t-\tau,\ \theta,\ \varphi)=$$
$$=\mathrm{Re}\left\{S(t-\tau)\exp\left[j2\pi f_0(t-\tau)+j\frac{\pi D}{\lambda}(2i-1)+j\varphi\right]\right\},$$
$$з_{2i}(t-\tau,\ \theta,\ \varphi)= \tag{7.117}$$
$$=\mathrm{Re}\left\{S(t-\tau)\exp\left[j2\pi f_0(t-\tau)-j\frac{\pi D}{\lambda}(2i-1)+j\varphi\right]\right\}.$$

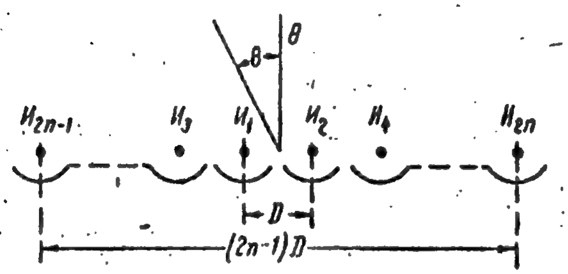

Fig. 7.8. Antenna arrangement for multichannel phase direction finding method

If the received signals $з_1$, $з_2$,..., $з_{2n}$ are processed directly in 2n independent receiving channels, the likelihood coefficient

$$\Lambda(\tau,\ \theta)=$$
$$=\exp(-2q^2n)I_0\left\{\frac{2}{N_0}L\left[\sum_{i=1}^{2n}\int x_i(t)\,з_i(t-\tau,\ \theta,\ \varphi)\,dt\right]\right\} \tag{7.118}$$

assuming that a signal with parameters $(\tau_0,\ \theta_0)$ is input, is

$$\Lambda(\tau,\ \theta)=\exp(-2q^2n)\,I_0\,[2n2q^2\,|\Psi(\tau-\tau_0)|\,|\Psi_T(\theta-\theta_0)|+ +N(\tau,\ \theta)], \tag{7.119}$$

where [30,74]

$$\Psi_T(\theta)=\frac{1}{n}\sum_{k=1}^{n}\cos\frac{\pi D}{\lambda}(2k-1)\theta=\frac{\sin 2n\frac{\pi D}{\lambda}\theta}{2n\sin\frac{\pi D}{\lambda}\theta}. \tag{7.120}$$

and the random process $N(\tau,\theta)$ is such that

$$\langle N(\tau_1, \theta_1) N(\tau_2, \theta_2)\rangle = 2n2q^2\Psi(\tau_1 - \tau_2)\Psi_\gamma(\theta_1 - \theta_2). \tag{7.121}$$

The angular coordinate correlation function (7.120) is periodic, with a repetition period of λ/D. Figure 7.9 shows the function $|\Psi_\gamma|$ in one repetition period for n=1 and n=4. The width of the base of the main lobe is λ/nD. The side lobes of the function are relatively small, and drop off as n becomes smaller.

The unambiguous detection angle in multichannel phase systems, which determines the working sector, thus does not depend upon the number of antennas, but rather the distance between adjacent antennas. On the other hand, the resolution depends upon the number of antennas and is determined by the overall extent of the antenna system 2nD. The number of resolvable angular coordinate positions can be assumed to be 2n.

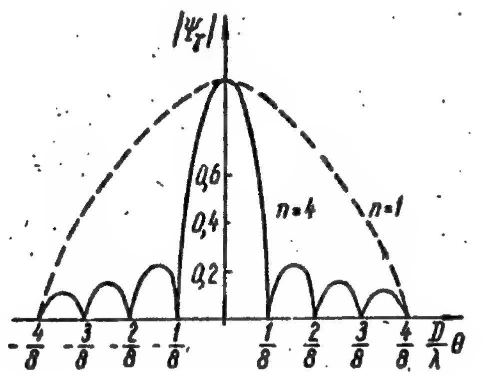

Fig. 7.9. Angular coordinate correlation function of signal for phase direction finding method (2n -- number of antennas).

We note that when the phase direction finding method is employed the analytical formulas for the angular coordinate performance indicators are similar to the formulas for the performance indicators for the range coordinates in Doppler systems. For example, the correlation functions $\Psi(\tau)$ of a Doppler system with 2n frequency components spaced 2F apart is

the same as the correlation function $\Psi_\gamma(\theta)$ of a phase direction finding system with 2n antenna systems spaced distance D apart (Fig. 7.8). The angle D/λ rad is the analog of the frequency interval of 2F.

The potential sampling accuracy of the angular coordinate is determined by formula (4.62), and is equal to

$$\delta_\theta^2 = \langle(\theta_0^* - \theta)^2\rangle = \frac{1}{2n2q^2\beta_\theta^2}, \tag{7.122}$$

where [30]

$$\beta_\theta^2 = |\Psi_\gamma''(0)| = \left(\frac{\pi D}{\lambda}\right)^2 \frac{1}{n} \sum_{k=1}^{n} (2k-1)^2 = \frac{4n^2-1}{3}\left(\frac{\pi D}{\lambda}\right)^2. \tag{7.123}$$

As the number of antennas, and accordingly the number of independent receiving channels, is increased the accuracy with which the angular coordinate is estimated increases monotonically. However, when the signal energy is fixed, like in the case of expanding the signal spectrum, the accuracy does not increase without limit, but only until the power ratio exceeds the threshold.

If the noise in the antenna feed system is significantly smaller than the total noise in the receiving channel, processing without forming sum and difference signals does not fully utilize the system capabilities. The loss increases as the number of channels 2n. Let us suppose that the signals з_{2i-1} and з_{2i}, received by the antennas in Fig. 7.8, are not input directly to 2n processing channels, but rather form 2n sum-and-difference outputs $\text{з}_{\Sigma 1}, \ldots \text{з}_{\Sigma 2n}$. In order to find the optimum nature of summation of the signals $\text{з}_1, \text{з}_2, \ldots, \text{з}_{2n}$, we shall examine the result of simple summation of the outputs of all of the antennas assuming that the signal has the parameters $(\tau_0, \theta_0, \phi_0)$

$$\sum_{i=1}^{2n} s_i(t-\tau_0, \theta_0, \varphi_0)=$$

$$=2\sum_{i=1}^{n} \mathrm{Re}\,\{S(t-\tau_0)\exp[j2\pi f_0(t-\tau_0)+j\varphi_0]\}\cos\left[\frac{\pi D}{\lambda}(2i-1)\theta_0\right]=$$

$$=2n\, s(t-\tau_0, 0, \varphi_0)\frac{\sin 2n\frac{\pi D}{\lambda}\theta_0}{2n\sin\frac{\pi D\theta_0}{\lambda}}. \qquad (7.124)$$

The summary output (7.124), which is applied to the input of one of the 2n processing channels, has the maximum possible intensity when $\theta_0=0$. It can be assumed that for any θ_0 belonging to the working sector $|\theta_0| \leq \lambda/2D$ the signal intensity will be close to the maximum possible value at the input of one of the 2n channels if the angular offset between the directions of maximum intensity in the adjacent channels is $\lambda/2nD$. In order to achieve this, the input to an arbitrary kth channel must be the signal

$$s_{\Sigma k}(t-\tau_0, \theta_0, \varphi_0)=$$

$$=2n\, s(t-\tau_0, 0, \varphi_0)\frac{\sin 2n\left[\frac{\pi D}{\lambda}\left(\theta_0+k\frac{\lambda}{2nD}\right)\right]}{2n\sin\left[\frac{\pi D}{\lambda}\left(\theta_0+k\frac{\lambda}{2nD}\right)\right]}, \qquad (7.125)$$

where $\quad k=-n, -n+1, \ldots, n-2, n-1.$

Considering (7.120) and (7.124), the expression for the sum-and-difference signal at the input to the kth channel can be changed as follows:

$$
\begin{aligned}
&s_{\Sigma k}(t-\tau_0, \theta_0, \varphi_0) = \\
&= \sum_{i=1}^{n} \cos\left[\pi \frac{k}{2n}(2i-1)\right][s_{2i}(t-\tau_0, \theta_0, \varphi_0) + s_{2i-1}(t-\tau_0, \theta_0, \varphi_0)] + \\
&+ \sin\left[\pi \frac{k}{2n}(2i-1)\right][s_{2i}(t-\tau_0, \theta_0, \varphi_0) - s_{2i-1}(t-\tau_0, \theta_0, \varphi_0)],
\end{aligned}
\tag{7.126}
$$

which indicates the manner in which the optimum sun-and-difference outputs are formed at the input of the 2n processing channels. If we disregard the occurrence of side lobes in function (7.120), the system becomes identical to the multichannel amplitude system whose directivity patterns are shown in Fig. 7.7. The width of the base of a directivity characteristic of each channel is λ/nD, and the angular offset between the characteristics of adjacent channels $\lambda/2nD$. The diagrams intersect at the 0.405 power level.

Finally, if a single processing channel is employed [73, 74], to which signals are applied alternately from different radiators with the amplitude method, or different sum-and-difference signals for the phase method, the system becomes equivalent to one which employs flat rocking of the directivity pattern. This again indicates that the potential capability of systems in scanning angular coordinates and direction finding are determined basically by the geometric dimensions of the antenna system, as well as the number of independent processing channels.

In order to simplify the task in this chapter we have excluded certain capabilities of antenna engineering. For example, we have assumed that the (voltage) directivity pattern $\gamma(\theta)$ of the antenna system is a real function of θ. Furthermore, the correlation function $\Psi_\gamma(\theta)$, which determines the potential capabilities of the system with respect to the angular coordinate θ, differs little from the function $\gamma(\theta)$. It obtains accordingly that the potential capabilities of the system are almost fully determined by the directivity pattern $\gamma(\theta)$ itself -- its width Δ. However, in the general case the function $\gamma(\theta)$ may be complex, and the correlation function $\Psi(\theta)$ may differ significantly in width from $\gamma(\theta)$. The relationship here is precisely the same as between the signal $S(t)$ and its correlation function $\Psi(\tau)$ [56].

Further, during our examination of phase (and other) systems we disregarded the time offset between the complex envelopes of the signal $S(t-\tau)$ received by different antenna elements. In accordance with the operating conditions which usually occur, the latter assumes that the signal correlation time τ_K is significantly greater than the time intervals ($\sim \frac{D \sin \theta}{c}$) corresponding to the difference in travel of the beam to

different antenna elements. However, we can imagine systems in which this condition is not met. Then the joint correlation function $\Psi(\tau;\theta)$ for the range coordinate τ and angular coordinate θ cannot be represented as the product $\Psi(\tau)\ \Psi_{\gamma}(\theta)$, and the performance indicators of the system in terms of the coordinates τ and θ become interrelated [56].

Finally, it would be helpful to examine some questions involved in the practical use of the phenomenon of superdirectivity [69].

Chapter 9. Supplements

9.1. Signal Reception Against Background of Interference with Unknown Intensity

Up to now we have been studying reception against the background of interference with fixed, a priori known intensity. Reception conditions such as these occur when the interference is caused by internal radio equipment noise. We shall now assume that the interference can be approximated by Gaussian white noise (§ 1.2) whose spectral intensity N_0 is unknown and is a random, but constant over each observation interval, quantity

$$N_0 = \eta \langle N_0 \rangle. \tag{9.1}$$

The interference intensity parameter η, according to definition (9.1), is the ratio of N_0 -- the spectral intensity of the realization of the interference in a given observation interval -- to the mathematical expectation of that quantity $\langle N_0 \rangle$. We shall also assume that the received signal $\varepsilon з(t-\tau,\ \phi,\ \Phi)$ is defined by formula (5.10). Then the likelihood coefficient for the ensemble of all possible values of the parameters τ and Φ can be expressed as

$$\Lambda(\tau, \Phi) = \frac{\int\limits_0^\infty p(\varepsilon)\, d\varepsilon \int\limits_0^\infty \frac{p(\eta)}{\eta^m} \exp\left(-\frac{Z^2 m + \varepsilon^2 q^2}{\eta}\right) I_0\left[\frac{\varepsilon}{\eta} Y(\tau, \Phi)\right] d\eta}{\int\limits_0^\infty \frac{p(\eta)}{\eta^m} \exp\left(-\frac{Z^2 m}{\eta}\right) d\eta}, \tag{9.2}$$

where $p(\varepsilon)$ and $p(\eta)$ are the distributions of the intensity parameters of the signal and interference,

$$Y(\tau, \Phi) = L\left[\frac{2}{\langle N_0\rangle}\int_{\tau}^{\tau+T} x(t)\, s(t-\tau, \varphi, \Phi)\, dt\right], \tag{9.3}$$

$$Z^2 = \frac{1}{m\langle N_0\rangle}\int_{\tau}^{\tau+T} x^2(t)\, dt. \tag{9.4}$$

Integration with respect to t in (9.3) and (9.4) must be done over the arbitrary interval which includes the reflected signal $s(t-\tau,\phi,\Phi)$. Assuming that the duration of the reflected signal is T, the integration interval will be assumed to be $(\tau, \tau + T)$. 2m is the number of measurements of the interference space, which is determined by the interference spectrum width f_m $(2m=2f_m T)$ and, according to the approximation employed for the noise background, is an unbounded large (practically very high) number. The energy signal/noise ratio q^2 is the ratio of the mathematical expectation of the signal energy to the mathematical expectation of the spectral intensity of the interference

$$q^2 = \frac{\int_0^T s^2(t, \varphi, \Phi)\, dt}{\langle N_0\rangle}. \tag{9.5}$$

The number Z^2, as will be shown later, is defined by (9.4) such that as $m \to \infty$ it has a finite limit.

We now re-write expression (9.2), transforming it somewhat:

$$\Lambda(\tau, \Phi) = \frac{\int_0^\infty p(\varepsilon)\, d\varepsilon \int_0^\infty p(\eta) \exp\left(-\frac{\varepsilon^2}{\eta} q^2\right) I_0\left[\frac{\varepsilon}{\eta} Y(\tau,\Phi)\right]\left[\frac{1}{\eta}\exp\left(-\frac{Z^2}{\eta}\right)\right]^m d\eta}{\int_0^\infty p(\eta)\left[\frac{1}{\eta}\exp\left(-\frac{Z^2}{\eta}\right)\right]^m d\eta}. \tag{9.6}$$

Asymptotic estimates of the integrals in the numerator and denominator of (9.6) can be obtained for large values of m by the saddle-point method [20]. The idea of this method in our case is based on the fact that the function

$$f(\eta)=\frac{1}{\eta}\exp\left(-\frac{Z^2}{\eta}\right), \tag{9.7}$$

which enters into both expressions beneath the integral, has a single clearly defined maximum at the point $\eta=Z^2$. The larger the value of the parameter m, the more clearly expressed is the maximum of the function $[f(\eta)]^m$. Therefore, for large m the main contribution to the value of the integrals comes from the vicinity of the point of the maximum $(Z^2-h,\ Z^2+h)$. The use of this method makes it possible to represent expression (9.6) as the ratio of the functions beneath the integral at the maximum point $(\eta=Z^2)$

$$\Lambda(\tau,\Phi)\sim\int\limits_0^\infty p(\varepsilon)\exp\left(-\frac{\varepsilon^2q^2}{Z^2}\right)I_0\left[\frac{\varepsilon Y(\tau,\Phi)}{Z^2}\right]d\varepsilon. \tag{9.8}$$

As m increases without limit the sign ~ of the asymptotic estimate becomes an equals sign.

By examining formula (9.8) we can make the following practical conclusions. The definition of an optimum system does not require knowledge of the a priori distribution of the interference intensity $p(\eta)$. The likelihood coefficient $\Lambda(\tau,\Phi)$, regardlees of the distribution of the signal intensity $p(\varepsilon)$ for every given value of the number Z^2, is a monotonically increasing function of the output $Y(\tau,\Phi)$. Therefore, the optimum decision making rule, which in the general case of the use of the Neumann-Pearson criterion is expressed by formulas (2.43) and (2.44), can be formulated as follows in our case. It is decided that the target is absent if for all possible τ and Φ

$$Y(\tau,\Phi)\leqslant Y_0(Z^2). \tag{9.9}$$

It is decided that a signal is present with parameters (τ_x, Φ_x) if

$$Y(\tau_x, \Phi_x) \geqslant Y_0(Z^2),$$
$$Y(\tau_x, \Phi_x) = \sup_{\tau, \Phi} [Y(\tau, \Phi)]. \tag{9.10}$$

The appearance of the function $Y_0(Z^2)$, which expresses the optimum relationship between the threshold level Y_0 and Z^2, is found from the equation

$$\Lambda_0 = \int_0^\infty p(\varepsilon) \exp\left(-\frac{\varepsilon^2 q^2}{Z^2}\right) I_0\left(\frac{\varepsilon Y_0}{Z^2}\right) d\varepsilon, \tag{9.11}$$

in which Λ_0 is assigned by the selected false alarm probability. On the basis of (9.11) we can write the optimum form of function $Y_0(Z^2)$ for the distributions $p(\varepsilon)$ ordinarily used. For a fixed signal intensity, when $p(\varepsilon)=\delta(\varepsilon-1)$,

$$\Lambda_0 = \exp\left(-\frac{q^2}{Z^2}\right) I_0\left(\frac{Y_0}{Z^2}\right)$$

and, considering that $q^2 >> Z^2$,

$$Y_0 \approx q^2 + cZ^2. \tag{9.12}$$

In the case of Rayleigh distribution of the signal intensities

$$\Lambda_0 = \frac{Z^2}{Z^2 + q^2} \exp\left[\frac{Y_0^2}{4Z^2(Z^2 + q^2)}\right]$$

and considering that $q^2 >> Z^2$,

$$Y_0 \approx cZ. \tag{9.13}$$

The constant coefficients c figuring in (9.12) and (9.13) are determined by the given value of Λ_0 or the given false alarm level F. The formula which connects the coefficient c in (9.13) with the probability F will be presented below.

One feature of decision making rule (9.9), (9.10) for systems with unknown interference intensity is the fact that the threshold level Y_0, like the output $Y(\tau,\phi)$, is a function of the input data x(t). Figure 9.1 shows an optimum processing circuit which follows directly from this decision rule. The circuit consists of two channels, the first of which is used to form the output $Y(\tau.\Phi)$. The structure of this channel was examined in sufficient detail in the preceding chapters. The second channel, which forms the output Z^2, incorporates linear system LS_2, which is a wideband (by comparison with the signal spectrum) amplifier, a square-law detector and a low-pass filter (LPF) with cutoff frequency of approximately 1/T. The filter provides integration over the interval $(\tau, \tau+T)$. The outputs $Y(\tau,\Phi)$ and Z^2 are input to the decision device, which forms the optimum threshold level $Y_0(Z^2)$ and makes the decision. There are some possible simplifications to the structure of the channel which forms the threshold level $Y_0(Z^2)$. In order to clarify these capabilities let us examine the analytical expression for the output Z^2. The number Z^2 represents the result of extended integration of the process $x^2(t)$, and is consequently approximately a normal random quantity.

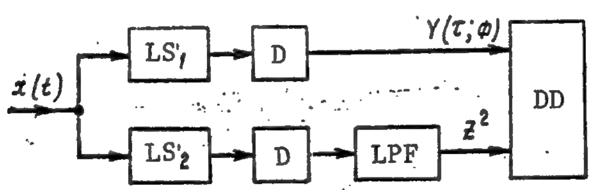

Fig. 9.1. Optimum processing circuit for signals received against background of interference with unknown intensity

The mathematical expectation and dispersion of the quantity Z^2 can be calculated by the formulas presented in § 1.2. In the case of the absence of a signal, when $x(t) = n(t)$, we obtain

$$Z^2 \approx \eta_0\left(1 + \theta\frac{1}{\sqrt{m}}\right), \tag{9.14}$$

where θ is a normalized normal random quantity, and η_0 is the realization of the noise intensity parameter η in the observation interval in question. In case the signal is present, when $x(t)=n(t)+\varepsilon_0 з(t-\tau,\phi,\Phi)$,

$$Z^2 \approx \eta_0\left(1 + \frac{\varepsilon_0^2}{\eta_0}\frac{q^2}{m} + \theta\frac{1}{\sqrt{m}}\right). \tag{9.15}$$

Assuming that the number

$$m = f_m T \tag{9.16}$$

is extremely large when the noise spectrum f_m is extremely wide, regardless of whether the reflected signal is present at the input, we can assume

$$Z^2 \approx \eta_0. \tag{9.17}$$

We now note that the output Z^2 of each channel in Fig. 9.1 retains its structure (9.14)-(9.17) regardless of the bandwidth Δf of the amplifier in each channel, if only $T\Delta f$ is large enough, e.g., of the order of 10^3 or more. In addition, the square-law detector can be replaced by a linear one, since

$$Z = \left[\frac{1}{(N_0)}\int_{\tau}^{\tau+T} x^2(t)\,dt\right]^{\frac{1}{2}} \approx \int_{\tau}^{\tau+T} L\left[\frac{1}{\sqrt{(N_0)}}\,x(t)\right]dt \approx \sqrt{\eta_0}. \tag{9.18}$$

Keeping these remarks in mind, the optimum processing circuit can be modified as shown in Fig. 9.2. Either a square-law or linear detector can be employed. This circuit assumes that a linear detector is used, and that the intensity of the reflected signals is described by a Rayleigh distribution. Furthermore, the appropriate choice of amplifier gain K ensures, in accordance with (9.13), that an optimum threshold level is formed.

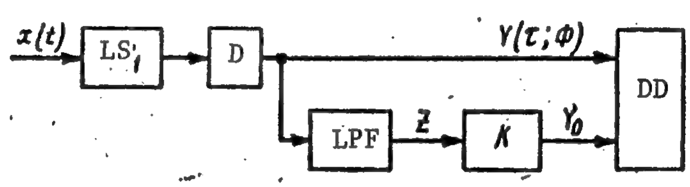

Fig. 9.2. Simplified optimum circuit for processing signals received against background of interference with unknown intensity

We shall now calculate the threshold ratio $q^2_{\text{пор}}$. We shall assume for simplicity that the useful signal parameters (τ, Φ) can take on only a single value (τ_0, Φ_0), i.e., we shall consider a two-alternative system. The false alarm probability F and detection probability D, keeping (9.17) in mind, are expressed as

$$F=\int_0^\infty p(\eta)\,d\eta \int_{Y_0(Z^\eta)}^\infty p_N(Y/\eta)\,dY, \tag{9.19}$$

$$D=\int_0^\infty p(\eta)\,d\eta \int_{Y_0(Z^\eta)}^\infty p_{sN}(Y/\eta)\,dY, \tag{9.20}$$

in which $p_N(Y/\eta)$ and $p_{sN}(Y/\eta)$ are the distribution functions of the output $Y=Y(\tau_0, \Phi_0)$ for the cases of the absence (N) and presence (SN) of the reflected signal, assuming that the spectral intensity of the interference is fixed and equal to $\eta\langle N_0\rangle$. We use the symbol σ^2 to denote the dispersion of the interference at the output of linear system LS_1

(Fig. 9.1 and 9.2), assuming that η=1. Then

$$p_N(Y/\eta)=\frac{Y}{\sigma^2\eta}\exp\left(-\frac{Y^2}{2\sigma^2\eta}\right) \tag{9.21}$$

and the false alarm probability for a given fixed value of the intensity parameter η is

$$F(\eta)=\int\limits_{Y_0(Z^2)}^{\infty} p_N(Y/\eta)\,dY=\exp\left[-\frac{Y_0^2(Z^2)}{2\sigma^2\eta}\right]. \tag{9.22}$$

The analysis below is limited to the most interesting case in which the signal intensity parameter ε is described by a Rayleigh distribution, and the optimum form of the function $Y_0(Z^2)$ is determined by formula (9.13). In this case

$$F=F(\eta)=\exp\left(-\frac{c^2}{2\sigma^2}\right), \tag{9.23}$$

whence it follows that the choice of the optimal form (9.13) of the function $Y_0(Z^2)$ ensures that the false alarm probability remains constant over observation intervals with different values of the realization of the quantity η. In addition, formula (9.23) makes it possible to use the required probability F to select the quantity c/σ, which defines the ratio of the gains in the channels which form $Y(\tau,\Phi)$ and Y_0.

When the distribution of the signal intensity parameter is described by a Rayleigh approximation

$$p_{s.N}(Y/\eta)=\frac{Y}{\sigma^2(\eta+q^2)}\exp\left[-\frac{Y^2}{2\sigma^2(\eta+q^2)}\right] \tag{9.24}$$

and the correct detection probability

$$D=\int\limits_0^{\infty} p(\eta)\,F^{\frac{\eta}{\eta+q^2}}\,d\eta \tag{9.25}$$

or

$$D \approx \int_0^{\infty} p(\eta) \exp\left[\frac{\eta}{q^2} \ln F\right] d\eta = 1 + \frac{\ln F}{q^2} + \sum_{k=2}^{\infty} \frac{\langle \eta^k \rangle}{k!} \left(\frac{\ln F}{q^2}\right)^k. \tag{9.26}$$

We can conclude from examining this expression that when the detection probabilities are high, on the order of 0.9 or more, regardless of the distribution of $p(\eta)$, and especially for a fixed interference intensity,

$$D \approx 1 - \frac{1}{q^2} \ln \frac{1}{F} \tag{9.27}$$

and the threshold ratio is

$$q^2_{\text{nop}} \approx \frac{\ln \frac{1}{F}}{1-D}. \tag{9.28}$$

For comparatively small values (of the order of 0.5 or less) the correct detection probability D, conversely, depends upon the distribution of the interference intensity $p(\eta)$.

We have investigated the characteristics of reception against the background of random interference with unknown intensity for signals of the type (5.10).* Obviously, the method used to calculate the likelihood coefficient can be extended to the case of receiving signals with other sets of useful and parasitic parameters. Like in the present case, the analytical expression for the likelihood coefficient for systems with unknown interference intensity is the same as the analytical expression for the likelihood coefficient in systems with fixed interference intensity, if we only replace N_0 in the latter with $Z^2\langle N_0\rangle$. Accordingly, by substituting $Z^2\langle N_0\rangle$ for N_0 we can extend all of the results and formulas of the theory of detection against the background of random interference with known intensity to the case of signal reception against the background of interference with unknown intensity.

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*[Chapter 5 not included in this translation. Tr.]

9.2. Use of Target Information Obtained During Previous Observation Intervals

All of our investigations have assumed that the duration of the reflected signals is small enough that the received oscillations contain information only about the instantaneous values of the current signal parameters. These signals, of comparatively short duration, were examined in isolation, i.e., disregarding the information contained in the signals reflected from the target during preceding time intervals. This statement of the problem corresponds to the actual operating conditions of modern radars. In this regard, Zibert [15] wrote, "We must go beyond the circle of ideal associated with finite (and consequently short) observation time.... Every radar target, generally speaking, has its own temporal and trajectory history, which is a more thorough expression of the distinguishing features of the target than the instantaneous values of its current coordinates and velocity. We must learn to use this target characteristic. More difficult theoretical, computational and practical problems arise in connection with this issue, but it is this area more than any other which holds hopes for future achievements in the development of radar engineering".

The practical solution of the problem of more or less total utilization of the target trajectory in order to increase radar operating range, accuracy and resolution is an extremely difficult matter. In the present section we shall suggest that a decision at an arbitrary moment in time T is made with allowance for the target information contained in the reflected signal for all of the preceding observation intervals, and we shall discuss in general features the extent to which this may influence the potential capabilities of the system in terms of range and detection accuracy. The extremely important and interesting question of the influence of this processing of the received signals on the resolution capability is not dealt with.

For definition, we shall be examining a periodic scanning detection and tracking system. After each scan cycle (or during each scan cycle) a statistical trial is made, as is a decision as to whether or not a target is present and concerning the instantaneous value of its current coordinates.

In the general case, target coordinates can be taken to mean an arbitrary set of useful parameters to be estimated. Information about the current coordinators of a detected target is produced after every scan cycle, so that the time variation of the set of useful parameters -- the target trajectory -- is determined practically unambiguously. The signal duration in each scan cycle is equal to the target irradiation time. The only difference of this system is the fact that a decision is made during each scan cycle on the basis of signals reflected from the target within the present cycle as well as all preceding cycles.

The signal reflected from a target, considering all of the scan cycles preceding the one in question, called the kth scan cycle, can in general be written as

$$s(t, \{s_i\}, \{u_i\}) = \sum_{i=1}^{k} s_i(t, s_i, u_i), \tag{9.29}$$

where s_i and u_i are the instantaneous values of the current useful and parasitic parameters, respectively, during the ith scan cycle. When a decision is made that an approaching target is present, the number k-1 of previous scan cycles taken into account, as will be shown below, is not important, since the energy (an accordingly the information) contained in the reflected signal for distant scan cycles is negligibly small. The multidimensional parasitic parameters u_i in different scan cycles will be considered independent random quantities which belong respectively to the regions U_i. The functions s_i with different values of the subscript i each have a different time offset by the corresponding number of scan periods. In addition, the average signal intensity varies in accordance with variation in the distance to the target during different scan periods.

Since we agreed not to deal with questions of resolution, the only useful effect provided by considering previous scan cycles is the increased signal energy realized by the receiving device when making a decision. We shall provide a quantitative estimate of this increase in the realized energy. We shall be using the following notation: R -- range, corresponding to the moment at which the decision is made, Q^2 -- energy of reflected signal for one scan cycle with target range of R, ΔR -- radial component of length of path covered by target during one scan cycle, α -- relative path length

$$\alpha = \frac{\Delta R}{R}. \tag{9.30}$$

For near-maximum ranges, the quantity α is small -- of the order of 0.01. Q_Σ^2 is the total energy of the reflected signal, considering all preceding scan cycles. The increase in realized energy can be estimated by the value of the ratio Q_Σ^2/Q^2. Then, assuming that the energy of the received signal

is inversely proportional to distance raised to the fourth power, we find

$$\frac{Q_\Sigma^2}{Q^2}=\sum_{k=0}^{\infty}\frac{1}{(1+k\alpha)^4}\approx\frac{1}{\alpha}\int_0^{\infty}\frac{dz}{(1+z)^4}=\frac{1}{3\alpha}. \tag{9.31}$$

The energy contained in sufficiently distant scan cycles, removed from the cycle in question, e.g., by more than $1/\alpha$ cycles, is negligibly small. In fact,

$$\sum_{k=0}^{\frac{1}{\alpha}}\frac{1}{(1+k\alpha)^4}\approx\frac{1}{\alpha}\int_0^{1}\frac{dz}{(1+z)^4}=\frac{0{,}875}{3\alpha}, \tag{9.32}$$

which differs little from (9.31)

The energy gain obtained by considering previous scan cycles thus increases as the distance to the target. For distances close to the maximum operating range, the total energy of the received signal over all of the scan cycles, or over the closest $1/\alpha$ cycles, is several tens of times greater than the signal energy during the last scan cycle. It can be expected that a sharp increase in energy will lead to a significant increase in range (or correct detection probability) and parameter estimation accuracy.

The second effect which must be dealt with when attempting to employ previous scan cycles in the decision making process is that the signal taken over several scan cycles contains more information, or a greater number of random parameters, than the signal over one cycle. Destruction or extraction of additional information requires the consumption of additional signal energy and, what is most important, it complicates significantly the optimum processing circuit. This aspect of the problem is dealt with in more detail below.

The a posteriori probability for a signal з with the set of useful parameters $\{s_i\} = s_1, s_2, \ldots, s_k$, on the basis of general principles, is

$$p(s_1,\ldots,s_k/x)=k_x p(s_1,\ldots,s_k)\Lambda(s_1,\ldots,s_k), \tag{9.33}$$

where the likelihood coefficient

$$\Lambda(s_1, \ldots, s_k) = \int\limits_{U_1} \ldots \int\limits_{U_k} p(u_1) \ldots p(u_k) \times \\ \times \exp\left\{-\frac{1}{N_0}\int\limits_{-\infty}^{\infty}\left[\sum_{l=1}^{k} s_l(t, s_l, u_l)\right]^2 dt + \right. \\ \left. + \frac{2}{N_0}\sum_{l=1}^{k}\int\limits_{-\infty}^{\infty} x(t)\, s_l(t, s_l, u_l)\, dt\right\} du_1 \ldots du_k. \tag{9.34}$$

The coefficient k_x denotes a number which depends only upon the received oscillation x(t), and which can have a different value in different formulas. The likelihood coefficient taken for one arbitrary ith scan cycle will be designated $\Lambda_i(s_i)$

$$\Lambda_l(s_l) = \int\limits_{U_l} p(u_l) \exp\left\{-\frac{1}{N_0}\int\limits_{-\infty}^{\infty} s_l^2(t, s_l, u_l)\, dt + \right. \\ \left. + \frac{2}{N_0}\int\limits_{-\infty}^{\infty} x(t)\, s_l(t, s_l, u_l)\, dt\right\} dt_l. \tag{9.35}$$

Then, considering that the functions s_i and s_j $(j \neq i)$ do not overlap in time, and also that the a priori probability $p(s_i, \ldots, s_k)$ can be written in the form

$$p(s_1, \ldots, s_k) = p(s_1, \ldots, s_{k-1})\, p(s_k / s_1, \ldots, s_{k-1}), \tag{9.36}$$

we find

$$\Lambda(s_1; \ldots, s_k) = \prod_{l=1}^{k} \Lambda_l(s_l) = \Lambda(s_1, \ldots, s_{k-1})\, \Lambda_k(s_k) \tag{9.37}$$

and

$$p(s_1,\ldots,s_k/x) = k_x p(s_1,\ldots,s_{k-1}/x)\, p(s_k/s_1,\ldots,s_{k-1})\, \Lambda_k(s_k). \tag{9.38}$$

The a posteriori probability $p(s_1,\ldots, s_k/x)$ is formed, consequently, by multiplying three functions: $p(s_1,\ldots, s_{k-1}/x)$ -- the a posteriori probability obtained at the end of the preceding k-1th scan cycle; $p(s_k/s_1,\ldots, s_{k-1})$ -- the a priori conditional probability of transition of the object to the point s_k after it has covered the trajectory $s_1,\ldots, s_{k-1}$, and $\Lambda_k(s_k)$ -- the likelihood coefficient corresponding to ordinary optimal processing of the received oscillation x(t) during the kth scan cycle taken in isolation. Processing of the received oscillation during the kth scan cycle, as follows from (9.38), can add nothing to the information about the target coordinates s_{k-1}, s_{k-2}, etc., obtained during preceding scan cycles. The purpose of forming the function $p(s_1,\ldots, s_k/x)$ is to ensure that the most reliable possible decision is made about the presence of the target and about its instantaneous coordinates s_k during the kth scan cycle. When the decision is made at the end of the kth cycle all of the other coordinates $s_{k-1}, s_{k-2},\ldots, s_1$ are not subject to estimation, and are parasitic random quantities. Accordingly, when k scan cycles are used in the decision making process, the optimum output should be the function

$$\begin{gathered}\hat{p}(s_k/x) = \int\limits_S \cdots \int\limits_S p(s_1,\ldots,s_k/x)\, ds_1 \ldots ds_{k-1} = \\ = k_x \Lambda_k(s_k) \left[\int\limits_S \cdots \int\limits_S p(s_1,\ldots,s_{k-1}/x)\, p(s_k/s_1,\ldots,s_{k-1})\, ds_1 \ldots ds_{k-1} \right]\end{gathered} \tag{9.39}$$

or any other function which corresponds one-to-one to (9.39). S is the region of variation of the useful parameter s_i.

The factor in the square brackets in the latter expression is the statistically substantiated prediction $\lambda(s_k)$ for the current scan cycle

$$\begin{gathered}\lambda(s_k) = \int\limits_S \cdots \int\limits_S p(s_1,\ldots,s_{k-1}/x)\, p(s_k/s_1,\ldots,s_{k-1})\, ds_1 \ldots ds_{k-1} = \\ = k_x \int\limits_S \cdots \int\limits_S \Lambda(s_1,\ldots,s_{k-1})\, p(s_1,\ldots,s_k)\, ds_1 \ldots ds_{k-1},\end{gathered} \tag{9.40}$$

obtained on the basis of observations over k-1 preceding cycles and on the basis of the a priori conditional probability $p(s_k/s_1, \ldots, s_{k-1})$. The function $\lambda(s_k)$ is the a priori probability distribution with respect to the kth scan cycle.

If the target coordinates s_i and s_{i+1} in adjacent scan cycles are totally connected random quantities, i.e., they are connected uniquely by a defined functional relationship,

$$s_1 = f_1(s_k), \ldots, s_{k-1} = f_{k-1}(s_k), \tag{9.41}$$

where $f_1, \ldots, f_{k-1}$ are certain uniquely defined functions.

Here

$$p(s_1, \ldots, s_k) = p(s_k)\,\delta[s_1 - f_1(s_k)] \ldots \delta[s_{k-1} - f_{k-1}(s_k)] \tag{9.42}$$

and

$$\lambda(s_k) = k_x p(s_k)\,\Lambda_1[f_1(s_k)] \ldots \Lambda_{k-1}[f_{k-1}(s_k)]. \tag{9.43}$$

Complete statistical connectedness of the target coordinates in different scan cycles can occur, for example, when all that is to be established is the presence of the target on a given trajectory, or for a practically stationary target. In the latter case

$$s_1 = s_2 = \ldots = s_k = s, \tag{9.44}$$

$$\lambda(s) = p(s)\,\Lambda_1(s) \ldots \Lambda_{k-1}(s) \tag{9.45}$$

$$\hat{p}(s/x) = k_x p(s)\,\Lambda_1(s) \ldots \Lambda_k(s), \tag{9.46}$$

which leads to the usual operation of incoherent accumulation of signal energy over k repetition periods, where the optimum output Y(s) can be the function

$$Y(s) = c \sum_{l=1}^{k} \ln \Lambda_l(s). \tag{9.47}$$

If the target coordinates in adjacent scan cycles, on the other hand, are statistically independent quantities, then

$$p(s_1, \ldots, s_k) = p(s_1) \ldots p(s_k), \tag{9.48}$$

$$\lambda(s_k) = p(s_k) \tag{9.49}$$

and

$$\hat{p}(s/x) = k_x p(s_k) \Lambda_k(s_k). \tag{9.50}$$

As might be expected, if the target coordinates are statistically independent in different scan cycles, no value can be extracted from the information obtained during the preceding cycles.

Thus, depending upon the degree of statistical connection between the target coordinates in adjacent scan cycles, the potential capabilities of a system which considers all preceding scan cycles approach one of two limits: the potential capabilities of a system with a single scan cycle and with signal energy Q^2, or the potential capabilities of a system with incoherent accumulation of approximately $1/\alpha$ repetition cycles and with total signal energy of Q_Σ^2. More detailed study of the question of the potential capabilities of systems employing previous scan cycles requires the use of statistical data concerning the possible target trajectories as the basis for studying the a priori joint distribution function $p(s_1, \ldots, s_k)$ of the sets of useful parameters $s_1, \ldots, s_i$ for $i=1,2,\ldots,k$. The absence of statistical data concerning the trajectories $s_1, \ldots, s_i$ of the reflecting objects makes it more difficult to discuss the question of resolution. It is clear, however, that the presence of the additional parameters $s_1, \ldots, s_{k-1}$ leads to additional capabilities of resolving received signals, while the averaging operation (9.39) destroys these capabilities.

Figure 9.3 shows one possible version of the block diagram of a system which makes complete use of the information over k scan cycles, which is a direct interpretation of analytical expression (9.39) for the a posteriori probability. The notation in the figure is as follows: C -- i-system which carries out operation (9.35) of converting the received oscillation x(t) to $\Lambda_i(s_i)$; x -- a multiplication device which multiplies the input oscillations (functions); $\int$ -- integrating device, which carries out operation (9.39) of eliminating the parameters $s_1, \ldots, s_{k-1}$ from the output. The outputs of the multiplying devices differ in several coefficients k_x for the functions $p(s_1, \ldots, s_i/x)$ shown in the figure, which, of course, does not violate the optimality of the system output.

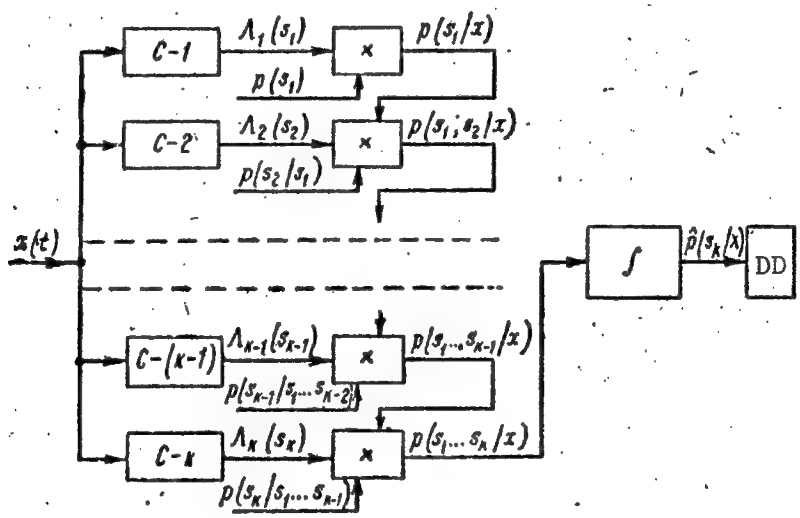

Fig. 9.3. Optimum signal processing circuit which realizes target trajectory information.

This optimum processing circuit is extremely complex. It can be simplified significantly if we avoid the attempt to reproduce exactly the optimum processing of the received oscillations over a large number of scan cycles. The gain achieved by considering previous scan periods will be smaller than the theoretical maximum, but it can still be sufficiently large. Investigation of these capabilities, as well as the a priori joint distributions for the sets of parameters $(s_1, \ldots, s_k)$ are beyond the scope of the present study.

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$$\left[\frac{\partial}{\partial \alpha} Y(\alpha, \beta)\right]_{\substack{\alpha=\alpha_\Phi \\ \beta=\beta_0^*}} = \varepsilon_0^* \sum_{n=1}^{\infty} 2n a_{2n,0}(\alpha_\Phi - \alpha_0^*)^{2n-1} \approx$$

$$\approx 2\varepsilon_0^* a_{20}(\alpha_\Phi - \alpha_0^*), \tag{4.68}$$

$$\left[\frac{\partial^2}{\partial \alpha^2} Y(\alpha, \beta)\right]_{\substack{\alpha=\alpha_\Phi \\ \beta=\beta_0^*}} = \varepsilon_0^* \sum_{n=1}^{\infty} 2n(2n-1) a_{2n,0}(\alpha_\Phi - \alpha_0^*)^{2n-2} \approx$$

$$\approx 2\varepsilon_0^* a_{20}.$$

System (4.68) can be solved for α_0*. For small errors and small deviations of the fixing point, when representation (4.35) is acceptable, we can use the approximate values of the right part of equations (4.68).

In this case

$$\alpha_0^* = \alpha_\Phi - \frac{Y'_\alpha(\alpha_\Phi, \beta_0^*)}{Y''_\alpha(\alpha_\Phi, \beta_0^*)}. \tag{4.69}$$

Thus, in order to obtain the best estimate α_0^* it is necessary to form the two numbers $Y'_\alpha(\alpha_\Phi, \beta_0)$ and $Y''_\alpha(\alpha_\Phi, \beta_0^*)$, the first of which represents the optimum error signal d.

By way of illustration, let us find the optimum scheme for forming the error signal and obtaining the estimate for the simplest signal $\varepsilon 3(t,\alpha)$ which satisfies condition (4.33) and contains the useful parameter α and parasitic parameter ε. According to (4.68) and (4.23)

$$d = Y'_\alpha(\alpha_\Phi) = \int_{-\infty}^{\infty} x(t) s'_\alpha(t, \alpha_\Phi)\, dt, \tag{4.70}$$